yields

$$\sqrt{2} = \frac{59}{20}\theta - \frac{1}{2}\theta^3 + \frac{1}{80}\theta^5,$$

$$\sqrt{3} = \frac{313}{80}\theta - \frac{297}{160}\theta^3 + \frac{67}{320}\theta^5 - \frac{3}{640}\theta^7,$$

$$\sqrt{7} = -\frac{469}{80}\theta + \frac{377}{160}\theta^3 - \frac{71}{320}\theta^5 + \frac{3}{640}\theta^7.$$

Finally, setting $t = \frac{\theta}{80} = \frac{\sqrt{2}+\sqrt{3}+\sqrt{7}}{80}$ we obtain $\sqrt{2} = f(t)$, $\sqrt{3} = g(t)$, and $\sqrt{7} = h(t)$, where the polynomials $f(x)$, $g(x)$, and $h(x)$ have integer coefficients :

$$f(x) = 236x - \frac{1}{2}(80)^3x^3 + (80)^4x^5,$$

$$g(x) = 313x - \frac{1}{2}(80)^2 \cdot 297x^3 + \frac{1}{4}(80)^4 \cdot 67x^5 - \frac{3}{8}(80)^6x^7,$$

$$h(x) = -469x + \frac{1}{2}(80)^2 \cdot 377x^3 - \frac{1}{4}(80)^4 \cdot 71x^5 + \frac{3}{8}(80)^6x^7.$$

Also solved by MOHAMMED AASSILA, Strasbourg, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNANDEZ, and LUZ RONCAL, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3307. [2008 : 45, 47] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Eliminate θ from the system

$$\lambda\cos(2\theta) = \cos(\theta+\alpha),$$

$$\lambda\sin(2\theta) = 2\sin(\theta+\alpha).$$

Similar solutions by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; Joe Howard, Portales, NM, USA; and George Tsapakidis, Agrinio, Greece.

The given system can be rewritten as a linear system in $\cos\alpha$ and $\sin\alpha$:

$$\cos\theta\cos\alpha - \sin\theta\sin\alpha = \lambda(\cos^2\theta - \sin^2\theta),$$

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = \lambda\sin\theta\cos\theta.$$

Its solution is then

$$\cos\alpha = \lambda\cos^3\theta, \quad \text{and} \quad \sin\alpha = \lambda\sin^3\theta;$$

whence,

$$(\cos\alpha)^{2/3} + (\sin\alpha)^{2/3} \;=\; \lambda^{2/3}\,. \qquad (1)$$

Comments from the Spanish team. Note that the original system has a solution if and only if α and λ satisfy (1). In particular, letting $x = \cos\alpha$ and $y = \sin\alpha$, we see that for $1 \le |\lambda| \le 2$ the solutions can be represented by the intersection points of the unit circle $x^2 + y^2 = 1$ with the astroid $x^{2/3} + y^{2/3} = \lambda^{2/3}$. Thus for each λ with absolute value between 1 and 2, (1) will be satisfied for eight values of α; for $\lambda \in \{\pm 1, \pm 2\}$, it will be satisfied by four values of α. There can be no real solutions for other values of λ.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; BOB SERKEY, Leonia, NJ, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect submission.

Our readers produced solutions in a variety of formats. Here are a few of the nicest. Instead of (1), Alt, Bataille, and the proposer independently obtained the equivalent equation

$$\sin^2(2\alpha) \;=\; \frac{4\left(\lambda^2-1\right)^3}{27\lambda^2}\,.$$

Geupel found that in terms of a real parameter t, the solutions of the given system satisfy

$$\theta \;=\; \arctan t + m\pi\,,\quad \alpha \;=\; \arctan(t^3) + n\pi\,,\quad \textit{and}\quad \lambda \;=\; (-1)^{m+n}\sqrt{\frac{(1+t^2)^3}{1+t^6}}\,,$$

for integers m and n. In addition, there were several implicit solutions where the solver simply presented an equation for θ in terms of α or λ; for example, $\theta = \arctan\sqrt[3]{\tan\alpha} + k\pi$ came from Arslanagić and from Ros.

3308. [2008 : 45, 48] *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Given $\triangle ABC$, let AD be the altitude to BC. If $AB : AC = 1 : \sqrt{3}$, prove that $AD \le \frac{\sqrt{3}}{2}BC$. When does equality hold?

I. *Solution by Joe Howard, Portales, NM, USA.*

It suffices to take $AB = 1$ and $AC = \sqrt{3}$; consequently, $AD = \sin B$. Writing $a = BC$, we therefore must show that

$$\sin B \;\le\; \frac{\sqrt{3}}{2}a\,.$$