SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Last year we received a batch of correct solutions from Steven Karp, student, University of Waterloo, Waterloo, ON, to problems 3289, 3292, 3294, 3296, 3297, 3298, and 3300, which did not make it into the December issue due to being misfiled. Our apologies for this oversight.

3301. [2008 : 44, 46] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\ln 2 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)}{n} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)}\,.$$

What is this common value ?

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain, expanded by the editor.

Let A and B denote the summations on the left side and the right side of the proposed equality, respectively. Also, let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &= H_{2n} - H_n \\ &= H_{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) = \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j}\,. \end{aligned}$$

Since it is well known that $\sum\limits_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} = \ln 2$, by changing the order of the double summation we have

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{j=2n+1}^{\infty} \frac{(-1)^{j-1}}{j} \right) = \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} \left(\sum_{n=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{1}{n} \right) \\ &= \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} = \sum_{k=1}^{\infty} H_k \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) \\ &= \sum_{k=1}^{\infty} \frac{H_k}{(2k+1)(2k+2)} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)} = B\,. \end{aligned}$$

To find the common value of the two absolutely convergent series, let

$$f(x) \;=\; \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} H_{\lfloor \frac{j-1}{2} \rfloor} x^j \,,$$

where the power series for f converges for all $x \in (-1,1)$. Then

$$\begin{aligned} f'(x) \;&=\; \sum_{j=3}^{\infty} (-1)^{j-1} H_{\lfloor \frac{j-1}{2} \rfloor} x^{j-1} \;=\; \sum_{j=2}^{\infty} (-1)^j H_{\lfloor \frac{j}{2} \rfloor} x^j \\ &=\; \sum_{n=1}^{\infty} \left(H_n x^{2n} - H_n x^{2n+1} \right) \;=\; (1-x) \sum_{n=1}^{\infty} H_n x^{2n} \,. \end{aligned} \tag{1}$$

Now, it is well known that

$$\frac{1}{1-x} \ln\left(\frac{1}{1-x}\right) \;=\; \sum_{n=1}^{\infty} H_n x^n \,. \tag{2}$$

[*Ed*: Multiply $\dfrac{1}{1-x} = 1+x+x^2+\cdots$ with $-\ln(1-x) = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$ and observe that the coefficient of x^n is $1+\frac{1}{2}+\cdots+\frac{1}{n}$.]

From (2) we have $(1-x) \sum\limits_{n=1}^{\infty} H_n x^n = -\ln(1-x)$.

Thus, $(1-x^2) \sum\limits_{n=1}^{\infty} H_n x^{2n} = -\ln(1-x^2)$, and it follows that

$$(1-x) \sum_{n=1}^{\infty} H_n x^{2n} \;=\; -\frac{1}{1+x} \ln(1-x^2) \,. \tag{3}$$

From (1) and (3) we obtain

$$f'(x) \;=\; -\frac{1}{1+x} \ln(1-x^2) \,.$$

Since $f(0) = 0$ and the last improper integral below is convergent, by applying Abel's Continuity Theorem for power series we have

$$A \;=\; \lim_{x \to 1^-} f(x) \;=\; \int_0^1 f'(x)\,dx \;=\; -\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \,. \tag{4}$$

It remains to evaluate the last integral in (4).

With the change of variable $x = 2u-1$, we have

$$\int_0^1 \frac{\ln\left(1-x^2\right)}{1+x}\,dx \;=\; \int_{1/2}^1 \frac{\ln\bigl(4u(1-u)\bigr)}{u}\,du \;=\; I_1 + I_2 \,, \tag{5}$$

where

$$I_1 = \int_{1/2}^{1} \frac{\ln(4u)}{u}\,du = \frac{1}{2}\ln^2(4u)\Big|_{1/2}^{1} = \frac{1}{2}\left((\ln 4)^2 - (\ln 2)^2\right)$$
$$= \frac{1}{2}(\ln 4 + \ln 2)(\ln 4 - \ln 2) = \frac{1}{2}(\ln 8)(\ln 2) = \frac{3}{2}(\ln 2)^2 . \quad (6)$$

On the other hand, using integration by parts and then making the change of variable $u = 1 - t$, we have

$$I_2 = \int_{1/2}^{1} \frac{\ln(1-u)}{u}\,du = (\ln u)\big(\ln(1-u)\big)\Big|_{1/2}^{1} + \int_{1/2}^{1} \frac{\ln u}{1-u}\,du$$
$$= -(\ln 2)^2 + \int_0^{1/2} \frac{\ln(1-t)}{t}\,dt = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt - I_2 ,$$

from which we obtain

$$2I_2 = -(\ln 2)^2 + \int_0^1 \frac{\ln(1-t)}{t}\,dt . \quad (7)$$

[Ed : Since the integrals in the computations above are improper, care must be taken ; e.g., the evaluation of $(\ln u)\big(\ln(1-u)\big)$ at $u = 1$ must be done by computing $\lim\limits_{u\to 1^-} (\ln u)\big(\ln(1-u)\big)$ using L'Hôspital's Rule.]

Finally,

$$\int_0^1 \frac{\ln(1-t)}{t}\,dt = -\int_0^1 \frac{1}{t}\ln\left(\frac{1}{1-t}\right)dt$$
$$= -\int_0^1 \frac{1}{t}\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\right)dt = -\sum_{n=1}^{\infty}\int_0^1 \frac{t^{n-1}}{n}\,dt$$
$$= -\sum_{n=1}^{\infty}\left(\frac{t^n}{n^2}\Big|_0^1\right) = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{6} . \quad (8)$$

From (4) – (8), we conclude that $A = B = \frac{\pi^2}{12} - (\ln 2)^2$.

Also solved by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy ; and the proposer. There were also three incomplete solutions, all of which only demonstrated that the two given summations are equal.

3302. [2008 : 44, 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of a triangle ABC, respectively. Show that

$$(s^2 + r^2 + 4Rr)(s^2 + r^2 + 2Rr) \geq 4Rr(5s^2 + r^2 + 4Rr) ,$$

and determine when equality holds.