therefore,

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \;\le\; \frac{1}{4}\sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right).$$

Using the relation $F_{2n+2} - F_{2n} = F_{2n+1}$ and the well known and easy to check formula $F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$, we have

$$\frac{1}{F_{2n+1}} \;=\; \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} \;=\; \frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}}{1 + \frac{1}{F_{2n}}\frac{1}{F_{2n+2}}},$$

and then

$$\begin{aligned}\arctan\left(\frac{1}{F_{2n+1}}\right) &= \arctan\left(\frac{\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}}{1 + \frac{1}{F_{2n}}\frac{1}{F_{2n+2}}}\right) \\ &= \arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{4}\sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right) &= \frac{1}{4}\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right)\right] \\ &= \frac{1}{4}\arctan\left(\frac{1}{F_2}\right) = \frac{1}{4}\arctan 1 = \frac{\pi}{16}.\end{aligned}$$

Thus, the sum of the given series does not exceed $\frac{\pi}{16} \approx 0.196$, which improves the proposed upper bound, because

$$\frac{4}{\pi}\arctan(\beta)\Big(\arctan(\beta) + \frac{1}{3}\Big) \approx 0.625.$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous also improved the proposed upper bound.

3262. [2007 : 299, 301] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let m be an integer, $m \ge 2$, and let $a_1, a_2, \ldots, a_m$ be positive real numbers. Evaluate the limit

$$L_m \;=\; \lim_{n\to\infty} \frac{1}{n^m}\int_1^e \prod_{k=1}^{m} \ln\big(1 + a_k x^n\big)\,dx.$$

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain, modified by the editor.

For each integer $m \geq 1$ we will show that

$$L_m = (-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k \frac{m!}{(m-k)!}. \tag{1}$$

First note that for $x \geq 1$, we have

$$xa_k^{1/n} \leq (1+a_kx^n)^{1/n} \leq x(1+a_k)^{1/n}. \tag{2}$$

Since $a_k^{1/n}$ and $(1+a_k)^{1/n}$ each converge to 1 as $n \to \infty$, it follows from the above that $(1+a_kx^n)^{1/n}$ converges to x as $n \to \infty$, thus,

$$\lim_{n\to\infty}\frac{\ln(1+a_kx^n)}{n} = \lim_{n\to\infty}\ln(1+a_kx^n)^{1/n} = \ln x. \tag{3}$$

Taking logarithms across the last inequality in (2), we obtain

$$\frac{\ln(1+a_kx^n)}{n} \leq \ln x + \frac{\ln(1+a_k)}{n} \leq \ln x + \ln(1+a_k),$$

from which it follows that

$$\prod_{k=1}^{m}\frac{\ln(1+a_kx^n)}{n} \leq \prod_{k=1}^{n}\big(\ln x + \ln(1+a_k)\big).$$

By Lebesgue's Dominated Convergence Theorem, we may bring the limit inside the integral; then we apply (3) as follows

$$\begin{aligned} L_m &= \int_1^e \lim_{n\to\infty}\prod_{k=1}^{n}\frac{\ln(1+a_kx^n)}{n}\,dx \\ &= \int_1^e \prod_{k=1}^{n}\lim_{n\to\infty}\frac{\ln(1+a_kx^n)}{n}\,dx \\ &= \int_1^e (\ln x)^m\,dx. \end{aligned} \tag{4}$$

Next we integrate by parts to derive the recurrence relation

$$L_m = e - mL_{m-1}. \tag{5}$$

Finally, we use induction on m to show that (with the appropriate initial condition) the solution to the reccurence in (5) is given by (1).

The case when $m = 1$ is clear, since the right side of (1) is 1 and from (4) we have $L_1 = \int_1^e \ln x\,dx = 1$.

Suppose (1) holds for some $m \geq 1$. Then using (5) we have

$$\begin{aligned} L_m &= e - m\left\{(-1)^m(m-1)! + e\sum_{k=0}^{m-1}(-1)^k\frac{(m-1)!}{(m-1-k)!}\right\} \\ &= e + (-1)^{m+1}m! + e\sum_{k=0}^{m-1}(-1)^{k+1}\frac{m!}{(m-1-k)!} \\ &= (-1)^{m+1}m! + e + e\sum_{k=1}^{m}(-1)^k\frac{m!}{(m-k)!} \\ &= (-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\,, \end{aligned}$$

and our proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was 1 incorrect solution submitted.

Janous notes the interesting fact that L_m can be expressed in terms of D_m, the number of derangements of $1, 2, \ldots, m$. (A permutation σ of $1, 2, \ldots, m$ is called a derangement *if $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, m$.) Since it is well known that $D_m = m!\sum\limits_{k=0}^{m}(-1)^k\frac{1}{k!}$, we see that $L_m = (-1)^{m+1}m! + (-1)^m eD_m$.*

The proposer remarked that his proposal was a generalization of the following problem which appeared in the Romanian journal Gazeta *in 2000:*

Compute $\lim\limits_{n\to\infty}\dfrac{1}{n^2}\displaystyle\int_1^e \ln(1+x^n)\ln(1+2x^n)\,dx$.

Both he and Bracken and Nadeau pointed out the interesting fact that the answer is completely independent of the a_k's given.