3261. [2007 : 299, 301] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{aligned} F_0 &= 0\,, \quad F_1 = 1\,, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}\,, \quad \text{for } n \geq 1; \\ L_0 &= 2\,, \quad L_1 = 1\,, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}\,, \quad \text{for } n \geq 1. \end{aligned}$$

Prove that

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \leq \frac{4}{\pi}\arctan(\beta)\left(\arctan(\beta)+\frac{1}{3}\right),$$

where $\beta = \frac{1}{2}(\sqrt{5}-1)$.

Solution by Manuel Benito, Óscar Ciaurri, Emilio Fernández, and Luz Roncal, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The following relations between the Fibonacci and Lucas numbers

$$L_{2n} + L_{2n+2} = 5F_{2n+1} \quad \text{and} \quad L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2\,,$$

are well known and easy to check. From these we have

$$\frac{1}{F_{2n+1}} = \frac{L_{2n}+L_{2n+2}}{L_{2n}L_{2n+2}-1} = \frac{\frac{1}{L_{2n}}+\frac{1}{L_{2n+2}}}{1-\frac{1}{L_{2n}}\frac{1}{L_{2n+2}}}\,,$$

so that

$$\begin{aligned} \arctan\left(\frac{1}{F_{2n+1}}\right) &= \arctan\left(\frac{\frac{1}{L_{2n}}+\frac{1}{L_{2n+2}}}{1-\frac{1}{L_{2n}}\frac{1}{L_{2n+2}}}\right) \\ &= \arctan\left(\frac{1}{L_{2n}}\right)+\arctan\left(\frac{1}{L_{2n+2}}\right). \end{aligned}$$

Applying the inequality $xy \leq \frac{1}{4}(x+y)^2$, we obtain

$$\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right) \leq \frac{1}{4}\left[\arctan\left(\frac{1}{F_{2n+1}}\right)\right]^2,$$

therefore,

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \;\leq\; \frac{1}{4}\sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right) .$$

Using the relation $F_{2n+2} - F_{2n} = F_{2n+1}$ and the well known and easy to check formula $F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$, we have

$$\frac{1}{F_{2n+1}} \;=\; \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} \;=\; \frac{\dfrac{1}{F_{2n}} - \dfrac{1}{F_{2n+2}}}{1 + \dfrac{1}{F_{2n}}\,\dfrac{1}{F_{2n+2}}} ,$$

and then

$$\begin{aligned}\arctan\left(\frac{1}{F_{2n+1}}\right) &= \arctan\left(\frac{\dfrac{1}{F_{2n}} - \dfrac{1}{F_{2n+2}}}{1 + \dfrac{1}{F_{2n}}\,\dfrac{1}{F_{2n+2}}}\right) \\ &= \arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right) .\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{4}\sum_{n=1}^{\infty}\arctan\left(\frac{1}{F_{2n+1}}\right) &= \frac{1}{4}\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{F_{2n}}\right) - \arctan\left(\frac{1}{F_{2n+2}}\right)\right] \\ &= \frac{1}{4}\arctan\left(\frac{1}{F_2}\right) = \frac{1}{4}\arctan 1 = \frac{\pi}{16} .\end{aligned}$$

Thus, the sum of the given series does not exceed $\frac{\pi}{16} \approx 0.196$, which improves the proposed upper bound, because

$$\frac{4}{\pi}\,\arctan(\beta)\Big(\arctan(\beta) + \frac{1}{3}\Big) \approx 0.625 .$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous also improved the proposed upper bound.

3262. [2007 : 299, 301] *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let m be an integer, $m \geq 2$, and let a_1, a_2, $\dots$, a_m be positive real numbers. Evaluate the limit

$$L_m \;=\; \lim_{n\to\infty} \frac{1}{n^m} \int_1^e \prod_{k=1}^{m} \ln\big(1 + a_k x^n\big)\, dx .$$