3253. [2007 : 297, 300] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \; < \; e^2 + \pi^2 \, .$$

I. *Essentially similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Tom Leong, Brooklyn, NY, USA; Andrea Munaro, student, University of Trento, Trento, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and the proposer.*

By the AM-GM Inequality, we have

$$\begin{aligned} \log_e(e^\pi - 1)\log_e(e^\pi + 1) &\le \left(\frac{\log_e(e^\pi - 1) + \log_e(e^\pi + 1)}{2}\right)^2 \\ &= \; \frac{1}{4}\Big(\log_e\big((e^\pi - 1)(e^\pi + 1)\big)\Big)^2 \; = \; \frac{1}{4}\big(\log_e(e^{2\pi} - 1)\big)^2 \\ &< \; \frac{1}{4}\big(\log_e(e^{2\pi})\big)^2 \; = \; \pi^2 \, . \end{aligned} \tag{1}$$

Similarly, we have

$$\log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \; < \; e^2 \, . \tag{2}$$

The result follows by adding (1) and (2).

II. *Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.*

For $a > 1$ and $b > 0$, we have

$$\begin{aligned} &\log_a(a^b - 1)\log_a(a^b + 1) \\ &\quad = \; \big(b + \log_a(1 - a^{-b})\big)\big(b + \log_a(1 + a^{-b})\big) \\ &\quad = \; b^2 + b\log_a(1 - a^{-2b}) + \log_a(1 - a^{-b})\log_a(1 + a^{-b}) \; < \; b^2 \, , \end{aligned}$$

since b and $1 - a^{-2b}$ are positive, $1 - a^{-b} < 1$, and $1 + a^{-b} > 1$. From the inequality above, we conclude that

$$\sum_{i=1}^{n} \log_{a_i}(a_i^{b_i} - 1)\log_{a_i}(a_i^{b_i} + 1) \; \le \; \sum_{i=1}^{n} b_i^2 \, ,$$

where $a_i > 1$ and $b_i > 0$ for $i = 1, 2, \ldots, n$.

The proposed inequality is the special case when $n = 2$, $a_1 = b_2 = e$ and $a_2 = b_1 = \pi$.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK, PAULA KOCA, and ANDREW SIEFKER, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France;

RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; ASHLEY TANGEMAN and PETRUS MARTINS, students, California State University, Fresno, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3254. [2007 : 298, 300] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a convex figure in the plane. A *diametrical chord* AB of $\mathcal{C}$ parallel to the direction vector $\overrightarrow{v}$ is a chord of $\mathcal{C}$ of maximal length parallel to the direction vector $\overrightarrow{v}$.

Prove that if every diametrical chord of $\mathcal{C}$ bisects the area enclosed by $\mathcal{C}$, then $\mathcal{C}$ must be centro-symmetric.

Solution by P.C. Hammer and T. Jefferson Smith from 1964, adapted by the editor.

What follows is a simplified version of the proof of Theorem 2.4 in [3]. In that work the authors prove that *any* convex planar body is centrally symmetric provided that each line bisecting the area is a diametral line. (Hammer and Smith use the words *diametral* and *centrally symmetric* rather than the equivalent but less common *diametrical* and *centro-symmetric*.)

Because in every direction there is exactly one line that bisects the given area, our assumption that every diametral chord is area bisecting implies that there is a unique diametral chord in every direction. The Hammer and Smith result is therefore stronger since it applies also to centrally symmetric regions whose boundary contains line segments (for which points of parallel sides are joined by parallel diametral chords, one of which bisects the area). In order to avoid a page of technical arguments, we will further restrict our result by assuming that the boundary of the convex region, denoted by $\mathcal{C}$, is a differentiable curve. For such boundaries our assumption that an area-bisecting chord is diametral implies that the tangent lines at its ends are parallel. We treat the plane as a vector space and let $u(\theta) = (\cos\theta, \sin\theta)$ be a unit vector function; then $u'(\theta) = (-\sin\theta, \cos\theta) = u(\theta + \frac{1}{2}\pi)$. Let $m(\theta)$ be the unique diametral line parallel to the direction of $u(\theta)$.

Then there exists a unique real number $p(\theta)$ such that $m(\theta)$ is representable as

$$\{x\colon\ x \cdot u'(\theta) = p(\theta)\} \ = \ \{x\colon\ x = p(\theta)u'(\theta) + tu(\theta),\ t \in \mathbb{R}\}\,. \quad (1)$$

Note that $p(\theta)u'(\theta)$ is the foot of the perpendicular from the origin to $m(\theta)$. Because we assume that $\mathcal{C}$ is differentiable, it follows easily that so is $p(\theta)$. We now represent $\mathcal{C}$ by a function $x(\theta)$ in the following way. Each line $m(\theta)$ intersects the boundary in two points, one of which is given by

$$x(\theta) \ = \ p(\theta)u'(\theta) \ + \ f(\theta)u(\theta)\,, \quad (2)$$