THE MULTIPLIER OF THE INTERVAL $[-1,1]$ FOR THE DUNKL TRANSFORM OF ARBITRARY ORDER ON THE REAL LINE

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Abstract. We study the boundedness of the multiplier of the interval $[-1,1]$ for the Dunkl transform of order $\alpha > -1$ on weighted L^p spaces, with $1 < p < \infty$. In particular, we get that it is bounded from $L^p(\mathbb{R}, |x|^{2\alpha+1}\,dx)$ into itself if and only if $4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$ when $\alpha > -1/2$ or if and only if $1 < p < \infty$ when $-1 < \alpha \le -1/2$.

1. Introduction and main result

For $\alpha > -1$, let J_α denote the Bessel function of order α:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}$$

(a classical reference on Bessel functions is [1]). Throughout this paper, by $\frac{J_\alpha(z)}{z^\alpha}$ we denote the even function

$$\frac{1}{2^\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}, \quad z \in \mathbb{C}.$$

In this way, for complex values of the variable z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}$$

(the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α). Moreover, let us take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(z), \quad z \in \mathbb{C}.$$

The Dunkl operators on $\mathbb{R}^n$ are differential-difference operators associated with some finite reflection groups (see [2]). We consider the Dunkl operator Λ_α, $\alpha \ge -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x} \left(\frac{f(x) - f(-x)}{2} \right). \tag{1.1}$$

For $\alpha \ge -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), & x \in \mathbb{R}, \\ f(0) = 1 \end{cases} \tag{1.2}$$

has $E_\alpha(\lambda x)$ as its unique solution (see [3] and [4]); this function is called the Dunkl kernel. For $\alpha = -1/2$, it is clear that $\Lambda_{-1/2} = d/dx$, and $E_{-1/2}(\lambda x) = e^{\lambda x}$.

THIS PAPER HAS BEEN PUBLISHED IN: *Integral Transforms Spec. Funct.* **26** (2015), 728–736, doi: `http://dx.doi.org/10.1080/10652469.2015.1038707`.

2000 *Mathematics Subject Classification.* Primary 42A45; Secondary 42A38.

Key words and phrases. Dunkl operator, Dunkl transform, Bessel functions, multipliers.

In a similar way to the Fourier transform (which is the particular case $\alpha = -1/2$), the Dunkl transform of order $\alpha \ge -1/2$ on the real line is

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} E_\alpha(-ixy) f(x)\, d\mu_\alpha(x), \qquad y \in \mathbb{R}, \tag{1.3}$$

were $d\mu_\alpha$ denotes the measure

$$d\mu_\alpha(x) = \tfrac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\, |x|^{2\alpha+1}\, dx.$$

The study of the properties of the Dunkl transform has a great interest, and many papers about this subject have been published during the last years. See, for instance, [5–14] and the references therein.

The behaviour of the Bessel functions is very well known. For instance, for real values of the variable, they verify the asymptotics

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha+1)} + O(x^{\alpha+2}), \qquad x \to 0^+; \tag{1.4}$$

and

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left[\cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to +\infty. \tag{1.5}$$

From these and similar results, and noticing that

$$E_\alpha(ix) = 2^\alpha \Gamma(\alpha+1)\, \frac{J_\alpha(x)}{x^\alpha} + 2^\alpha \Gamma(\alpha+1)\, \frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, xi, \tag{1.6}$$

it is easy to check that $|E_\alpha(ix)| \le 1$ for every $x \in \mathbb{R}$. Then, (1.3) is well defined for every $f \in L^1(\mathbb{R}, d\mu_\alpha)$, and

$$\|\mathcal{F}_\alpha f\|_{\infty,\alpha} \le \|f\|_{1,\alpha}, \tag{1.7}$$

where we use $\|\cdot\|_{p,\alpha}$ as a shorthand for $\|\cdot\|_{L^p(\mathbb{R}, d\mu_\alpha)}$.

Just as the Fourier transform, $\mathcal{F}_\alpha$ is an isomorphism of the Schwartz class S into itself, and $\mathcal{F}_\alpha^2 f(x) = f(-x)$. Fubini's theorem implies the multiplication formula

$$\int_{\mathbb{R}} \mathcal{F}_\alpha f(x) g(x)\, d\mu_\alpha(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha g(x) f(x)\, d\mu_\alpha(x), \quad f, g \in S.$$

Taking $g(x) = \overline{\mathcal{F}_\alpha f(x)}$ we get $\|\mathcal{F}_\alpha f\|_{2,\alpha} = \|f\|_{2,\alpha}$, $f \in S$. By density, this can be extended to functions in $L^2(\mathbb{R}, d\mu_\alpha)$.

Via S, the $[-1,1]$-multiplier $\mathcal{M}_\alpha$ is defined as

$$\mathcal{M}_\alpha f(x) = \mathcal{F}_\alpha(\chi_{[-1,1]} \mathcal{F}_\alpha f)(-x) \tag{1.8}$$

or, equivalently,

$$\mathcal{F}_\alpha(\mathcal{M}_\alpha f)(x) = \chi_{[-1,1]}(x) \mathcal{F}_\alpha f(x),$$

which is the usual notation.

In the paper [15], the authors studied the boundedness of the operator $\mathcal{M}_\alpha$ for $\alpha \ge -1/2$ in weighted L^p spaces.

But the Dunkl transform $\mathcal{F}_\alpha$ can also be defined in $L^2(\mathbb{R}, d\mu_\alpha)$ for $\alpha > -1$, although some properties like (1.7) are no longer valid for $-1 < \alpha < -1/2$. However, it preserves the same properties in $L^2(\mathbb{R}, d\mu_\alpha)$; see [16] for details. The aim of this paper is to extend the case $\alpha \ge -1/2$ in [15] to the whole range $\alpha > -1$.

Thus, the main result of this paper is the following:

Theorem. *Let $\alpha > -1$, $1 < p < \infty$, and $w_{a,b}(x) = |x|^a (1+|x|)^{b-a}$. Then, there exists a constant C such that*

$$\|w_{a,b} \mathcal{M}_\alpha f\|_{p,\alpha} \le C \|w_{a,b} f\|_{p,\alpha} \tag{1.9}$$

if and only if

$$-\frac{2\alpha+2}{p} < a < (2\alpha+2)\left(1-\frac{1}{p}\right) \tag{1.10}$$

and

$$\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p} < b < -\frac{2\alpha+1}{2} + (2\alpha+2)\left(1-\frac{1}{p}\right). \tag{1.11}$$

As a simple consequence, it is easy to check that, in the *unweighted* case $a = b = 0$, we have

$$\|\mathcal{M}_\alpha f\|_{p,\alpha} \le C\|f\|_{p,\alpha} \iff \begin{cases} \dfrac{4(\alpha+1)}{2\alpha+3} < p < \dfrac{4(\alpha+1)}{2\alpha+1}, & \text{if } \alpha \ge -1/2, \\ 1 < p < \infty, & \text{if } -1 < \alpha < -1/2. \end{cases}$$

In Figure 1 we show the region of boundedness of the multiplier $\mathcal{M}_\alpha$ for $\alpha > -1$ and $1 < p < \infty$; there, the filled region consists of the values $(\alpha, 1/p)$ such that $\mathcal{M}_\alpha$ is bounded from $L^p(\mathbb{R}, d\mu_\alpha)$ into itself.

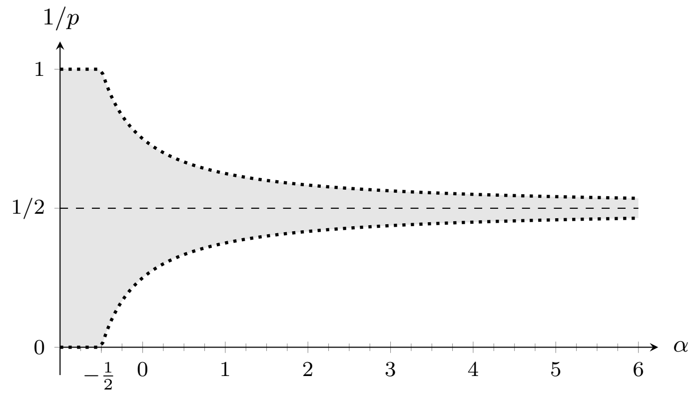

FIGURE 1. Region of boundedness of $\mathcal{M}_\alpha$.

Although the theorem presented in this paper is very similar to the theorem in [15], the proof is not the same. It is interesting to explain the difference.

The usual procedure in harmonic analysis to study the boundedness of an operator on L^p spaces is to write it as an integral operator with a kernel and then analyse such kernel. For our multiplier $\mathcal{M}_\alpha$, by using (1.8) and (1.3), we have

$$\begin{aligned} \mathcal{M}_\alpha f(x) &= \mathcal{F}_\alpha(\chi_{[-1,1]}(r)\mathcal{F}_\alpha f(r))(-x) = \int_{-1}^{1} E_\alpha(irx)\mathcal{F}_\alpha f(r)\,d\mu_\alpha(r) \\ &= \int_{-1}^{1} E_\alpha(irx)\left(\int_{\mathbb{R}} E_\alpha(-iyr)f(y)\,d\mu_\alpha(y)\right)d\mu_\alpha(r) \\ &= \int_{\mathbb{R}}\left(\int_{-1}^{1} E_\alpha(irx)E_\alpha(-iry)\,d\mu_\alpha(r)\right)f(y)\,d\mu_\alpha(y), \end{aligned}$$

where in the last step we have used Fubini's theorem (which is justified for suitable functions and extended in the usual way). Then, $\mathcal{M}_\alpha$ can be written as

$$\mathcal{M}_\alpha f(x) = \int_{\mathbb{R}} \mathcal{K}_\alpha(x,y)f(y)\,d\mu_\alpha(y) \tag{1.12}$$

with kernel

$$\mathcal{K}_\alpha(x,y) = \int_{-1}^{1} E_\alpha(irx)E_\alpha(-iry)\,d\mu_\alpha(r). \tag{1.13}$$

The next step is to get a good expression for the kernel. In [15] we see that, for $x, y \in \mathbb{R}$, $x \neq y$, one has

(1.14)
$$\int_{-1}^{1} E_\alpha(ixr)E_\alpha(-iyr)\,d\mu_\alpha(r) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\frac{E_\alpha(ix)E_\alpha(-iy) - E_\alpha(-ix)E_\alpha(iy)}{i(x-y)}$$

(this expression has been useful not only in [15], but also in [6] in connection with a sampling theorem related to the Dunkl transform).

Then, the operator $\mathcal{M}_\alpha$ can be written as

$$\begin{aligned}(1.15)\quad \mathcal{M}_\alpha f(x) &= \int_{\mathbb{R}} \mathcal{K}_\alpha(x,y)f(y)\,d\mu_\alpha(y)\\ &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\frac{E_\alpha(ix)E_\alpha(-iy)}{i(x-y)}\,f(y)\,d\mu_\alpha(y)\\ &\qquad - \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\frac{E_\alpha(-ix)E_\alpha(iy)}{i(x-y)}\,f(y)\,d\mu_\alpha(y)\\ &= \frac{1}{2^{2\alpha+2}\Gamma(\alpha+1)^2}\left(\mathcal{T}_\alpha^1 f(x) - \mathcal{T}_\alpha^2 f(x)\right),\end{aligned}$$

with

$$\mathcal{T}_\alpha^1 f(x) = E_\alpha(ix)H\big((E_\alpha(-iy)/i)f(y)|y|^{2\alpha+1}\big)(x),$$
$$\mathcal{T}_\alpha^2 f(x) = E_\alpha(-ix)H\big((E_\alpha(iy)/i)f(y)|y|^{2\alpha+1}\big)(x),$$

where H denotes the Hilbert transform.

In [15], the process continues by proving the L^p-boundedness of the operators $\mathcal{T}_\alpha^1$ and $\mathcal{T}_\alpha^2$ by means of the A_p theory of weights, what holds for $\alpha \geq -1/2$ in the corresponding range of p's.

But this method is not longer valid for the case $-1 < \alpha < -1/2$ because, if we try to follow it, we find that neither $\mathcal{T}_\alpha^1$ nor $\mathcal{T}_\alpha^2$ are bounded operators in the requires L^p spaces. Therefore, such decomposition is not useful in the range $-1 < \alpha < -1/2$.

Thus, in this paper we will use a different decomposition of the kernel, that will lead to another decomposition of $\mathcal{M}_\alpha$ as the sum of two operators. Then, we will make a clever use of the A_p theory of weights to prove the L^p-boundedness of these new operators. Note that the decomposition shown in this paper works both for the cases $\alpha \geq -1/2$ and $-1 < \alpha < -1/2$, so the present proof also covers the result previously stated in [15].

2. Proof of the theorem

The method to prove that the conditions (1.10) and (1.11) are necessary for (1.9) is as in [15]; here, we will not repeat it. Then, let us see that, provided $\alpha > -1$ and $1 < p < \infty$, the conditions (1.10) and (1.11) are sufficient for (1.9).

Given $p \in (1, \infty)$, a weight w in $\mathbb{R}$ is said to belong to the A_p class if

$$\left(\int_I w(x)\,dx\right)\left(\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C|I|^p$$

for every interval $I \subseteq \mathbb{R}$, with C independent of I. An important application of A_p theory lies on its relation with the boundedness of the Hilbert transform

$$Hg(x) = \int_{\mathbb{R}} \frac{g(y)}{x-y}\,dy.$$

Indeed, in [17] (see also [18] for further information) it is proved that

$$H\colon L^p(\mathbb{R}, w) \to L^p(\mathbb{R}, w) \text{ bounded} \iff w \in A_p.$$

For radial weights, it is well known that

$$|x|^\beta \in A_p \iff -1 < \beta < p-1.$$

If we take $\Phi(x) = |x|^r$ for $x \in [-1,1]$ and $\Phi(x) = |x|^s$ for $x \in (-\infty,1) \cup (1,\infty)$,

$$\Phi \in A_p \iff -1 < r < p-1 \text{ and } -1 < s < p-1 \tag{2.1}$$

(the intuitive behaviour is clear; see [19] for details and a proof).

In what follows, we will use C (perhaps with subindex) to denote a positive constant independent of x or f (and all other variables), which can assume different values in different occurrences. Moreover, for non-negative functions u and v defined on an interval, $u(x) \sim v(x)$ means that there exist two positive constants C_1 and C_2 such that $C_1 \le u(x)/v(x) \le C_2$.

As explained in the introduction, we are going to find a decomposition of $\mathcal{M}_\alpha f$ different to (1.15). To do this, we will manipulate the right hand side in (1.14). By substituting (1.6), we obtain

$$\begin{aligned} E_\alpha(ix)&E_\alpha(-iy) - E_\alpha(-ix)E_\alpha(iy) \\ &= 2^{2\alpha+1}\Gamma(\alpha+1)^2 \left(\frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, ix\, \frac{J_\alpha(y)}{y^\alpha} - \frac{J_\alpha(x)}{x^\alpha}\, \frac{J_{\alpha+1}(y)}{y^{\alpha+1}}\, iy \right) \end{aligned}$$

and then the kernel can be expressed, for $x \neq y$, as

$$\int_{-1}^1 E_\alpha(ixr)E_\alpha(-iyr)\, d\mu_\alpha(r) = 2^\alpha \Gamma(\alpha+1) \frac{\frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, x\, \frac{J_\alpha(y)}{y^\alpha} - \frac{J_\alpha(x)}{x^\alpha}\, \frac{J_{\alpha+1}(y)}{y^{\alpha+1}}\, y}{x-y}. \tag{2.2}$$

Now, to write $\mathcal{M}_\alpha f$ in terms of Hilbert transforms to apply the A_p theory of weights, let us use (1.12), (1.13) and (2.2). In this way, we get

$$\begin{aligned} \mathcal{M}_\alpha f(x) &= \int_{\mathbb{R}} \mathcal{K}_\alpha(x,y) f(y)\, d\mu_\alpha(y) \\ &= 2^\alpha\Gamma(\alpha+1) \int_{\mathbb{R}} \frac{\frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, x\, \frac{J_\alpha(y)}{y^\alpha}}{x-y} f(y)\, d\mu_\alpha(y) \\ &\qquad - 2^\alpha\Gamma(\alpha+1) \int_{\mathbb{R}} \frac{\frac{J_\alpha(x)}{x^\alpha}\, \frac{J_{\alpha+1}(y)}{y^{\alpha+1}}\, y}{x-y} f(y)\, d\mu_\alpha(y) \\ &= \mathcal{U}_\alpha^1 f(x) - \mathcal{U}_\alpha^2 f(x), \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} \mathcal{U}_\alpha^1 f(x) &= x\, \frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, H\left(\frac{J_\alpha(y)}{y^\alpha} f(y) |y|^{2\alpha+1} \right)(x), \\ \mathcal{U}_\alpha^2 f(x) &= \frac{J_\alpha(x)}{x^\alpha}\, H\left(y\, \frac{J_{\alpha+1}(y)}{y^{\alpha+1}}\, f(y)|y|^{2\alpha+1} \right)(x). \end{aligned}$$

With this decomposition, (1.9) follows if we prove that there exists a constant C independent of $f \in L^p(\mathbb{R}, w_{a,b}(x)^p |x|^{2\alpha+1}\, dx)$ such that

$$\|\mathcal{U}_\alpha^j f(x) w_{a,b}(x)\|_{L^p(\mathbb{R},|x|^{2\alpha+1}\,dx)} \le C \|f(x) w_{a,b}(x)\|_{L^p(\mathbb{R},|x|^{2\alpha+1}\,dx)}, \quad j = 1,2.$$

Taking $g(y) = (J_\alpha(y)/y^\alpha) f(y) |y|^{2\alpha+1}$, the inequality corresponding to $j = 1$ is equivalent to

$$\begin{aligned} &\|Hg(x)\|_{L^p(\mathbb{R},|xJ_{\alpha+1}(x)/x^{\alpha+1}|^p w_{a,b}(x)^p |x|^{2\alpha+1}\,dx)} \\ &\qquad \le C\|g(x)\|_{L^p(\mathbb{R},|J_\alpha(x)/x^\alpha|^{-p}|x|^{-(2\alpha+1)p} w_{a,b}(x)^p |x|^{2\alpha+1}\,dx)}; \end{aligned} \tag{2.4}$$

similarly, the corresponding to $j = 2$ is equivalent to

$$\begin{aligned} &\|Hg(x)\|_{L^p(\mathbb{R},|J_\alpha(x)/x^\alpha|^p w_{a,b}(x)^p |x|^{-(2\alpha+1)p}|x|^{2\alpha+1}\,dx)} \\ &\qquad \le C\|g(x)\|_{L^p(\mathbb{R},|xJ_{\alpha+1}(x)/x^{\alpha+1}|^{-p}|x|^{-(2\alpha+1)p} w_{a,b}(x)^p |x|^{2\alpha+1}\,dx)}. \end{aligned} \tag{2.5}$$

Let us start with (2.4). It is enough to prove that there is a weight $\Phi \in A_p$ with

$$C_1 \left| x\, \frac{J_{\alpha+1}(x)}{x^{\alpha+1}} \right|^p w_{a,b}(x)^p |x|^{2\alpha+1} \le \Phi(x) \le C_2 \left| \frac{J_\alpha(x)}{x^\alpha} \right|^{-p} |x|^{-(2\alpha+1)p} w_{a,b}(x)^p |x|^{2\alpha+1}.$$

Remember that we are here using the even functions $J_\alpha(x)/x^\alpha$ and $J_{\alpha+1}(x)/x^{\alpha+1}$ defined on the whole real line, so we can write this condition as

$$\begin{aligned} &C_1 |J_{\alpha+1}(|x|)|^p |x|^{-\alpha p} w_{a,b}(x)^p |x|^{2\alpha+1} \\ &\qquad \le \Phi(x) \le C_2 |J_\alpha(|x|)|^{-p} |x|^{-(\alpha+1)p} w_{a,b}(x)^p |x|^{2\alpha+1}. \end{aligned} \tag{2.6}$$

We have $w_{a,b}(x) \sim |x|^a$ in $[-1,1]$ and $w_{a,b}(x) \sim |x|^b$ in $(-\infty,-1] \cup [1,\infty)$. Moreover, from the estimates (1.4) and (1.5) it is clear that, for $\alpha > -1$,

$$|J_\alpha(|x|)| \le \begin{cases} C_\alpha |x|^\alpha, & \text{if } x \in [-1,1], \\ C_\alpha |x|^{-1/2}, & \text{if } x \in (-\infty,-1] \cup [1,\infty), \end{cases}$$

with a C_α constant depending only on α. According to these bounds, we have

$$|J_{\alpha+1}(|x|)|^p |x|^{-\alpha p} w_{a,b}(x)^p |x|^{2\alpha+1} \le \begin{cases} C|x|^{p+ap+2\alpha+1}, & \text{if } |x| \in (0,1), \\ C|x|^{-p/2-\alpha p+bp+2\alpha+1}, & \text{if } |x| \in (1,\infty), \end{cases}$$

and

$$|J_\alpha(|x|)|^{-p} |x|^{-(\alpha+1)p} w_{a,b}(x)^p |x|^{2\alpha+1} \ge \begin{cases} C|x|^{-(2\alpha+1)p+ap+2\alpha+1}, & \text{if } |x| \in (0,1), \\ C|x|^{-p/2-\alpha p+bp+2\alpha+1}, & \text{if } |x| \in (1,\infty). \end{cases}$$

Let us write

$$\Phi(x) = \begin{cases} |x|^r, & \text{if } |x| \in (0,1), \\ |x|^{-p/2-\alpha p+bp+2\alpha+1}, & \text{if } |x| \in (1,\infty). \end{cases}$$

By (2.1), $\Phi \in A_p$ will hold and fulfil (2.6) if

$$\begin{cases} -(2\alpha+1)p + ap + 2\alpha + 1 \le r \le p + ap + 2\alpha + 1, \\ -1 < r < p-1, \\ -1 < -p/2 - \alpha p + bp + 2\alpha + 1 < p - 1. \end{cases} \tag{2.7}$$

The third line is equivalent to

$$\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p} < b < -\frac{2\alpha+1}{2} + (2\alpha+2)\left(1 - \frac{1}{p}\right),$$

which is the condition (1.11) of the theorem. For the inequalities in (2.7) involving r, let us first note that $-(2\alpha+1)p + ap + 2\alpha + 1 \le p + ap + 2\alpha + 1$ is equivalent to $0 \le (2\alpha+2)p$, and this is true for any $\alpha > -1$. Then, for the existence of r it is enough to show that $-(2\alpha+1)p + ap + 2\alpha + 1 < p - 1$ and $-1 < p + ap + 2\alpha + 1$. Even more, the existence of r will be also guaranteed if we substitute the condition $-1 < p + ap + 2\alpha + 1$ by $-1 < ap + 2\alpha + 1$, that is stronger. Then, it suffices to note that the inequalities $-(2\alpha+1)p + ap + 2\alpha + 1 < p - 1$ and $-1 < ap + 2\alpha + 1$ are equivalent to

$$-\frac{2\alpha+2}{p} < a < (2\alpha+2)\left(1 - \frac{1}{p}\right),$$

that is the condition (1.10) of the theorem.

The study of (2.5) is completely similar. Instead of (2.7) we get the conditions

$$\begin{cases} -2(\alpha+1)p + ap + 2\alpha + 1 \le r \le ap + 2\alpha + 1, \\ -1 < r < p-1, \\ -1 < -p/2 - \alpha p + bp + 2\alpha + 1 < p - 1, \end{cases} \tag{2.8}$$

and, now, the inequality $-2(\alpha+1)p+ap+2\alpha+1 \le ap+2\alpha+1$ is true for any $\alpha > -1$. With small variations of the arguments used for $\mathcal{U}_\alpha^1$ we deduce the boundedness of $\mathcal{U}_\alpha^2$, and consequently the theorem is proved.

Remark. If we try to follow the method in [15] with the operators $\mathcal{T}_\alpha^1$ and $\mathcal{T}_\alpha^2$ in the decomposition (1.15), the first lines in (2.7) and (2.8) become, instead, $-(2\alpha+1)p+ap+2\alpha+1 \le r \le p+ap+2\alpha+1$ (twice the same condition). But, this time, the inequality $-(2\alpha+1)p+ap+2\alpha+1 \le ap+2\alpha+1$ is not true in general for any $\alpha > -1$ (only for $\alpha \ge -1/2$), so the method fails (actually, it is possible to prove that neither $\mathcal{T}_\alpha^1$ nor $\mathcal{T}_\alpha^2$ are bounded operators in the whole range given by (1.10) and (1.11) when $-1 < \alpha < -1/2$, but this is not of interest).

Funding. The research of the authors is supported by the DGI (Ministerio de Economía y Competitividad, Spain) under Grant MTM2012-36732-C03-02.

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