ON SHARP HEAT AND SUBORDINATED KERNEL ESTIMATES IN THE FOURIER-BESSEL SETTING

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ABSTRACT. We prove qualitatively sharp heat kernel bounds in the setting of Fourier-Bessel expansions when the associated type parameter ν is half-integer. Moreover, still for half-integer ν, we also obtain sharp estimates of all kernels subordinated to the heat kernel. Analogous estimates for general $\nu > -1$ are conjectured. Some consequences concerning the related heat semigroup maximal operator are discussed.

1. Introduction. Let J_ν denote the Bessel function of the first kind and order ν, and let $\{\lambda_{n,\nu} : n \geq 1\}$ be the sequence of successive positive zeros of J_ν in increasing order. It is well known that, for each fixed $\nu > -1$, the functions

$$\phi_n^\nu(x) = d_{n,\nu}\, x^{-\nu-1/2}(\lambda_{n,\nu}x)^{1/2} J_\nu(\lambda_{n,\nu}x), \quad n = 1, 2, \ldots,$$

form an orthonormal basis in $L^2((0,1), d\mu_\nu)$; here

$$d_{n,\nu} = \sqrt{2}|\lambda_{n,\nu}^{1/2} J_{\nu+1}(\lambda_{n,\nu})|^{-1}$$

are normalizing constants, and the measure is given by

$$d\mu_\nu(x) = x^{2\nu+1}\, dx.$$

The system $\{\phi_n^\nu : n = 1, 2, \ldots\}$ is usually referred to as the Fourier-Bessel system. Another Fourier-Bessel system arises by considering the functions

$$\psi_n^\nu(x) = x^{\nu+1/2}\phi_n^\nu(x), \quad n = 1, 2, \ldots,$$

which form an orthonormal basis in $L^2((0,1), dx)$. Fourier-Bessel expansions are of interest and have existed in the literature for a long time, see [32, Chapter XVIII]. The related convergence problems were investigated in [2, 3, 33]. For the study of several fundamental harmonic analysis operators in the Fourier-Bessel context, see [10, 11, 13, 12, 14, 15, 23] and references therein. Some interesting open questions concerning harmonic analysis of Fourier-Bessel expansions can be found in a recent paper by Betancor [4].

Given $\nu > -1$, consider the differential operator

$$L_\nu = -\Delta - \frac{2\nu + 1}{x} \frac{d}{dx},$$

which is symmetric and nonnegative on $C_c^2(0,1) \subset L^2((0,1), d\mu_\nu)$. Each ϕ_n^ν, $n = 1, 2, \ldots$, is an eigenfunction of L_ν with the corresponding eigenvalue $\lambda_{n,\nu}^2$,

$$L_\nu \phi_n^\nu = \lambda_{n,\nu}^2 \phi_n^\nu.$$

This leads to a natural self-adjoint extension of L_ν given by

$$\mathcal{L}_\nu f = \sum_{n=1}^\infty \lambda_{n,\nu}^2 \langle f, \phi_n^\nu \rangle_{d\mu_\nu} \, \phi_n^\nu$$

on the domain Dom $\mathcal{L}_\nu$ consisting of all $f \in L^2((0,1), d\mu_\nu)$ for which the series converges in $L^2((0,1), d\mu_\nu)$; here $\langle f, \phi_n^\nu \rangle_{d\mu_\nu}$ is the nth Fourier-Bessel coefficient of f. Clearly, the spectral decomposition of $\mathcal{L}_\nu$ is given by the ϕ_n^ν.

The aim of this paper is to find sharp estimates for the kernels

$$(1) \qquad G_t^{\nu,\alpha}(x,y) = \sum_{n=1}^\infty \exp\left(-t\lambda_{n,\nu}^\alpha\right) \phi_n^\nu(x) \phi_n^\nu(y),$$

where $x, y \in (0,1)$, $t > 0$ and $\alpha \in (0,2]$ is the index of subordination. These are precisely the integral kernels of the semigroups

$$\{T_t^{\nu,\alpha}\}_{t>0} := \left\{\exp(-t\mathcal{L}_\nu^{\alpha/2})\right\}_{t>0}$$

generated either by $\mathcal{L}_\nu$ itself ($\alpha = 2$) or by its fractional powers ($\alpha < 2$). Needless to say, the cases $\alpha = 2, 1$ are of prior importance since then (1) becomes the heat or the Poisson kernel, respectively, related to system $\{\phi_n^\nu\}$. It is well known, see for instance [30], that heat and

Poisson semigroups are powerful tools for developing harmonic analysis of operators. In fact, in the problems posed in [**4**], heat and Poisson semigroups play an essential role, and in this connection it is very desirable to have a sharp description for the corresponding integral kernels.

The behavior of $G_t^{\nu,\alpha}(x,y)$ does not seem to have been studied before, except for some rough estimates for the Poisson kernel $G_t^{\nu,1}(x,y)$ that can be found in [**12,** Proposition 6]. However, they are only from above, and the way to get them is relatively sophisticated by performing an involved analysis of the related series. Besides, such estimates do not reveal the interplay between t, x and y. Note that, for short times t, a direct analytic treatment of the series (1) is a complicated matter except for a few particular cases discussed in Section 4. First of all, the series defining $G_t^{\nu,\alpha}(x,y)$ is highly oscillating, and its behavior is hidden behind subtle cancelations between the oscillations. Moreover, J_ν is in general a transcendental function, and there is no explicit formula for its zeros. Finally, the power α of the eigenvalues in the argument of the exponential makes an even more sophisticated situation. These obstacles can be better understood by means of the explicit computations in Section 4 for the cases $\nu = \pm 1/2$, $\alpha = 1, 2$.

Our method of estimating $G_t^{\nu,\alpha}(x,y)$ is based on a connection between the Fourier-Bessel context and the classical setting related to multi-dimensional Euclidean balls. More precisely, we show that the fundamental solution $\mathcal{G}_t^{(d),2}(\mathbf{x},\mathbf{y})$ of the classical heat equation in the d-dimensional unit ball, subject to the Dirichlet boundary condition, is related to $G_t^{\nu,2}(x,y)$ with $\nu = d/2 - 1$. This allows us to transfer known bounds for $\mathcal{G}_t^{(d),2}(\mathbf{x},\mathbf{y})$ to the Fourier-Bessel framework on the interval $(0,1)$ and conclude qualitatively sharp estimates for $G_t^{\nu,2}(x,y)$ when ν is half-integer, see Theorem 3.3. Therefore, the method of transference enables us to obtain sharp estimates for the heat kernel in the Fourier-Bessel setting in a relatively straightforward manner. Essentially, the same procedure applies to subordinated kernels $\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y})$ and $G_t^{\nu,\alpha}(x,y)$, see Theorem 3.4. For large values of t, sharp estimates of $G_t^{\nu,\alpha}(x,y)$ can be derived directly from series representation (1), thanks to exponential decay. In this case all $\nu > -1$ are covered, see Theorem 3.7. Notice that all the results can be immediately translated to the setting of system $\{\psi_n^\nu\}$ since the relevant kernels coincide up to the factor $(xy)^{\nu+1/2}$.

We emphasize that the sharp bounds for $\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y})$, $0 < \alpha \le 2$, are quite strong results and seem to have been obtained in a complete form only recently [**29, 34**]; this may be a bit surprising, in view of simplicity of the geometry of the underlying domain, the Euclidean ball. On the other hand, a part of these results has been established on a probabilistic background. It seems that the connection between them and the problem we investigate in the present paper, posed as a harmonic analysis problem, has not been properly noticed before.

The paper is organized as follows. In Section 2 we first discuss the connection between the Fourier-Bessel setting in the interval $(0,1)$ and the situation associated with Euclidean balls and then relate the kernels $G_t^{\nu,\alpha}(x,y)$ and $\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y})$. In Section 3 we invoke necessary bounds for $\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y})$ and transfer them to the Fourier-Bessel setting in the interval. We also establish the long time behavior of $G_t^{\nu,\alpha}(x,y)$ by analyzing the defining series. Finally, Section 4 is devoted to various comments and remarks. These concern, in particular, mapping properties of the maximal operators of the semigroups $\{T_t^{\nu,\alpha}\}_{t>0}$, $0 < \alpha \le 2$.

Throughout the paper, we use a standard notation. While writing estimates, we will use the notation $X \lesssim Y$ to indicate that $X \le CY$ with a positive constant C independent of significant quantities. We shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$.

2. Connection with multi-dimensional balls. For $d \ge 1$, let $B^d = \{\mathbf{x} \in \mathbf{R}^d : |\mathbf{x}| < 1\}$ be the unit ball in $\mathbf{R}^d$ and denote $S^{d-1} = \partial B^d$. It is well known (see for instance [**19,** Chapter 2, H]) that there exists an orthonormal basis of eigenfunctions associated with the Dirichlet Laplacian in B^d. These are expressed explicitly by the functions ϕ_n^ν and spherical harmonics, as described below. For more details on spherical harmonics and their connections with symmetry properties of the Fourier transform we refer to [**31,** Chapter IV].

Given $k \ge 0$, let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k in $\mathbf{R}^d$. Taking harmonic polynomials in $\mathcal{P}_k$ and restricting them to S^{d-1} we obtain the space

$$H_k = \{P|_{S^{d-1}} : P \in \mathcal{P}_k \quad \text{and} \quad \Delta P = 0\}.$$

The elements of H_k are called spherical harmonics of degree k. The

dimension of H_k is finite; in fact, we have

$$\mathfrak{d}_k = \dim H_k = (2k + d - 2)\frac{(k + d - 3)!}{k!(d - 2)!}, \quad d \geq 2.$$

The case $d = 1$ is degenerate: S^0 consists of two points and so $\mathfrak{d}_0 = \mathfrak{d}_1 = 1$ and $\mathfrak{d}_k = 0$ for $k > 1$.

Let $\{Y_m^k : 1 \leq m \leq \mathfrak{d}_k\}$ be an orthonormal basis for H_k in $L^2(S^{d-1}, d\sigma)$, where σ is the standard (non-normalized) surface area measure on S^{d-1}. Define the functions

$$\Phi_{n,k,m}^{(d)}(\mathbf{x}) = \phi_n^{k+d/2-1}(|\mathbf{x}|)Y_m^k\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right),$$

$$n \geq 1, \quad k \geq 0, \ 1 \leq m \leq \mathfrak{d}_k;$$

here and later on, we tacitly assume that for $d = 1$ only $k = 0, 1$ are considered. The system

$$\left\{\Phi_{n,k,m}^{(d)} : n \geq 1, k \geq 0, 1 \leq m \leq \mathfrak{d}_k\right\}$$

is orthonormal and complete in $L^2(B^d, d\mathbf{x})$. Moreover, it consists of eigenfunctions of the Dirichlet Laplacian in B^d,

$$\Delta\Phi_{n,k,m}^{(d)} = -\lambda_{n,k+d/2-1}^2\Phi_{n,k,m}^{(d)}, \quad \Phi_{n,k,m}^{(d)}\Big|_{S^{d-1}} = 0.$$

Thus, the associated heat kernel and the subordinated kernels are expressed, for $\mathbf{x}, \mathbf{y} \in B^d$, $t > 0$, as

$$\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y}) = \sum_{n \geq 1}\sum_{k \geq 0}\sum_{1 \leq m \leq \mathfrak{d}_k} \exp\left(-t\lambda_{n,k+d/2-1}^\alpha\right)\Phi_{n,k,m}^{(d)}(\mathbf{x})\Phi_{n,k,m}^{(d)}(\mathbf{y}),$$

where $\alpha \in (0, 2]$ is the subordination index. These are the integral kernels of the semigroups

$$\left\{\mathcal{T}_t^{(d),\alpha}\right\}_{t>0} := \left\{\exp\left(-t(-\Delta)^{\alpha/2}\right)\right\}_{t>0},$$

where $-\Delta$ is understood to be the nonnegative self-adjoint operator in $L^2(B^d, d\mathbf{x})$ whose spectral resolution is given by the $\Phi_{n,k,m}^{(d)}$. The domain of this operator can be identified with the Sobolev space $H_0^1(B^d)$, cf., [1].

We now observe that the analysis of the radial case in the context of expansions with respect to $\{\Phi_{n,k,m}^{(d)}\}$ reduces to the Fourier-Bessel

setting in the interval $(0,1)$ with the type index $\nu = d/2 - 1$. Indeed, if $F(\mathbf{x}) = f(|\mathbf{x}|)$ is radial, then integrating in polar coordinates and taking into account that $Y_1^0 \equiv (\sigma(S^{d-1}))^{-1/2}$ we see that

$$
\begin{aligned}
\langle F, \Phi_{n,0,1}^{(d)} \rangle &= \int_{B^d} F(\mathbf{x}) \Phi_{n,0,1}^{(d)}(\mathbf{x})\, d\mathbf{x} = c_d \int_0^1 f(x) \phi_n^{d/2-1}(x)\, x^{d-1}\, dx \\
&= c_d \langle f, \phi_n^{d/2-1} \rangle_{d\mu_{d/2-1}},
\end{aligned}
$$

where $c_d = (\sigma(S^{d-1}))^{1/2}$. On the other hand, another integration in polar coordinates shows that, for $k > 0$, the Fourier coefficients $\langle F, \Phi_{n,k,m}^{(d)} \rangle$ vanish since then $Y_m^k \perp Y_1^0 \equiv \text{const.}$ Thus, the expansion of F in B^d is in fact the expansion of its profile f with respect to the system $\{\phi_n^{d/2-1}\}$ on $(0,1)$. Similarly, the associated heat and subordinated semigroups are also related via the radial case, as stated below.

Proposition 2.1. *Let $d \geq 1$, $\nu = d/2 - 1$ and $f \in \text{span}\{\phi_n^\nu : n \geq 1\}$. Then*

$$
\left(T_t^{\nu,\alpha} f\right) \circ \varphi(\mathbf{x}) = \mathcal{T}_t^{(d),\alpha}(f \circ \varphi)(\mathbf{x}), \quad \mathbf{x} \in B^d,
$$

where $\varphi(\mathbf{x}) = |\mathbf{x}|$.

Proof. For $f = \phi_n^\nu$, we have

$$
\begin{aligned}
\left(T_t^{\nu,\alpha} \phi_n^\nu\right) \circ \varphi(\mathbf{x}) &= e^{-t\lambda_{n,\nu}^\alpha} \phi_n^\nu(|\mathbf{x}|) = c_d e^{-t\lambda_{n,d/2-1}^\alpha} \Phi_{n,0,1}^{(d)}(\mathbf{x}) \\
&= c_d \mathcal{T}_t^{(d),\alpha} \Phi_{n,0,1}^{(d)}(\mathbf{x}) = \mathcal{T}_t^{(d),\alpha}\left(\phi_n^\nu \circ \varphi\right)(\mathbf{x}),
\end{aligned}
$$

where $c_d = (\sigma(S^{d-1}))^{1/2}$. The conclusion follows. $\square$

In fact the semigroups are related in the same way for more general functions f. This is confirmed by the relation between the corresponding integral kernels established below.

Theorem 2.2. *Let $d \geq 1$ and $\nu = d/2 - 1$. Then*

$$
G_t^{\nu,\alpha}(x,y) = \int_{S^{d-1}} \mathcal{G}_t^{(d),\alpha}(\mathbf{x}, y\xi)\, d\sigma(\xi), \quad x,y \in (0,1),\ t > 0,
$$

where $\mathbf{x} = (x, 0, \ldots, 0) \in B^d$.

Proof. Let $f \in \mathrm{span}\{\phi_n^\nu\}$. Using Proposition 2.1 and then, integrating in polar coordinates, we get

$$
\int_0^1 G_t^{\nu,\alpha}(|\mathbf{x}|,y) f(y) y^{2\nu+1} \, dy = \int_{B^d} \mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y}) f(|\mathbf{y}|) \, dy
$$
$$
= \int_0^1 \int_{S^{d-1}} \mathcal{G}_t^{(d),\alpha}(\mathbf{x},y\xi) \, d\sigma(\xi) \, f(y) y^{d-1} \, dy.
$$

Since the subspace spanned by the ϕ_n^ν is dense in $L^2((0,1),d\mu_\nu)$ we conclude the desired identity up to an exceptional set of y of measure 0. This set, however, must be empty because the kernels involved are continuous with respect to their arguments. $\qquad\square$

Actually, each of the kernels $G_t^{\nu,\alpha}(x,y)$ and $\mathcal{G}_t^{(d),\alpha}(\mathbf{x},\mathbf{y})$ is a jointly smooth function of $t > 0$ and its arguments. This can be verified even directly, by term-by-term differentiation of the defining series, with the aid of basic bounds for $d_{n,\nu}$, $\lambda_{n,\nu}$ and J_ν (see [**14,** Section 2]) and the fact that $|Y_m^k(\xi)|$ can be dominated, uniformly in m and ξ, by a polynomial in k (see for instance [**31,** Chapter IV, Corollary 2.9 (b)]). We leave details to interested readers.

3. Kernel estimates. Heat kernels in various contexts were extensively investigated in the literature. In particular, it is well known that the heat kernel corresponding to the Dirichlet Laplacian in B^d, $d \geq 1$, satisfies the bounds (cf., [**17,** (1.9.1)])

$$
(2) \qquad 0 < \mathcal{G}_t^{(d),2}(\mathbf{x},\mathbf{y}) \leq \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right),
$$

uniformly in $\mathbf{x},\mathbf{y} \in B^d$ and $t > 0$. These bounds are perhaps most clear from the probabilistic point of view since $\mathcal{G}_t^{(d),2}(\mathbf{x},\mathbf{y})$ is just the transition probability density of the time-scaled Brownian motion B_{2t} killed upon leaving B^d. However, finding sharp bounds for $\mathcal{G}_t^{(d),2}(\mathbf{x},\mathbf{y})$ that describe precisely the interplay between $\mathbf{x},\mathbf{y}$ and t, and the boundary behavior, is a much more complicated matter. The complete and qualitatively sharp estimates are available only recently, for bounded $C^{1,1}$ domains. The case of B^d, obviously being geometrically simpler, does not seem to have been known earlier.

Theorem 3.1 ([**16, 18, 29, 34**]). *Let $d \geq 1$. Given $T > 0$, there exists a constant $c > 1$ such that*

(3)
$$\left[\frac{(1 - |\mathbf{x}|)(1 - |\mathbf{y}|)}{t} \wedge 1 \right] \frac{1}{t^{d/2}} \exp \left(-\frac{c|\mathbf{x} - \mathbf{y}|^2}{t} \right)$$
$$\lesssim \mathcal{G}_t^{(d),2}(\mathbf{x}, \mathbf{y}) \lesssim$$
$$\left[\frac{(1 - |\mathbf{x}|)(1 - |\mathbf{y}|)}{t} \wedge 1 \right] \frac{1}{t^{d/2}} \exp \left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{ct} \right),$$

uniformly in $\mathbf{x}, \mathbf{y} \in B^d$ and $0 < t \leq T$. Moreover,

(4)
$$\mathcal{G}_t^{(d),2}(\mathbf{x}, \mathbf{y}) \simeq (1 - |\mathbf{x}|)(1 - |\mathbf{y}|) \exp \left(-t\lambda_{1,d/2-1}^2 \right),$$

uniformly in $\mathbf{x}, \mathbf{y} \in B^d$ and $t \geq T$.

The upper bound in (3) as well as the large time behavior (4) has been known at least since the 1980s, cf., Davies [**16**] and Davies and Simon [**18**], see also the related comments in [**34**]. The existence of the lower bound in (3) was shown by Zhang [**34,** Theorem 1.1] under the assumption $d \geq 3$ and with an implicitly fixed T. Later these restrictions were removed by Song [**29,** Theorems 3.8 and 3.9]. Note that the upper estimate in (3) holds in fact for all $t > 0$, which is not the case of the lower bound.

Sharp estimates for the subordinated kernels were found recently by Song [**29**].

Theorem 3.2 ([**29,** Theorem 4.7]). *Let $d \geq 1$ and $0 < \alpha < 2$. Given $T > 0$, we have*

(5)
$$\mathcal{G}_t^{(d),\alpha}(\mathbf{x}, \mathbf{y}) \simeq \left[\frac{(1 - |\mathbf{x}|)(1 - |\mathbf{y}|)}{t^{2/\alpha} + |\mathbf{x} - \mathbf{y}|^2} \wedge 1 \right] \frac{t}{(t^{2/\alpha} + |\mathbf{x} - \mathbf{y}|^2)^{(d+\alpha)/2}},$$

uniformly in $\mathbf{x}, \mathbf{y} \in B^d$ and $0 < t \leq T$.

Similarly as in (3), the upper bound in (5) holds in fact for all $t > 0$, which is not the case of the lower bound. The expected large time

behavior for $0 < \alpha < 2$ is of course

$$\mathcal{G}_t^{(d),\alpha}(\mathbf{x}, \mathbf{y}) \simeq (1 - |\mathbf{x}|)(1 - |\mathbf{y}|) \exp\left(-t\lambda_{1,d/2-1}^\alpha\right),$$

$$\mathbf{x}, \mathbf{y} \in B^d, \quad t \geq T.$$

When T is chosen sufficiently large, these bounds can be proved by a direct analysis of the series defining $\mathcal{G}_t^{(d),\alpha}(\mathbf{x}, \mathbf{y})$, similar to that in the proof of Theorem 3.7 below. This, however, is beyond the scope of this paper.

We now transfer the above bounds for $\mathcal{G}_t^{(d),\alpha}(\mathbf{x}, \mathbf{y})$ to the Fourier-Bessel setting on the interval $(0, 1)$. More precisely, we shall prove the following.

Theorem 3.3. *Let $\nu = d/2 - 1$ for some $d \geq 1$. Given $T > 0$, there exists a constant $c > 1$ such that*

$$(xy)^{-\nu-1/2}\left(\frac{xy}{t} \wedge 1\right)^{\nu+1/2}\left[\frac{(1-x)(1-y)}{t} \wedge 1\right]\frac{1}{\sqrt{t}}\exp\left(-c\frac{(x-y)^2}{t}\right)$$

$$\lesssim G_t^{\nu,2}(x,y) \lesssim$$

$$(xy)^{-\nu-1/2}\left(\frac{xy}{t} \wedge 1\right)^{\nu+1/2}\left[\frac{(1-x)(1-y)}{t} \wedge 1\right]\frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{ct}\right),$$

uniformly in $x, y \in (0, 1)$ and $0 < t \leq T$. Moreover,

$$G_t^{\nu,2}(x,y) \simeq (1-x)(1-y)\exp\left(-t\lambda_{1,\nu}^2\right), \quad x, y \in (0,1), \ t \geq T.$$

Theorem 3.4. *Let $\alpha \in (0, 2)$ and $\nu = d/2 - 1$ for some $d \geq 1$. Given $T > 0$, we have*

$$G_t^{\nu,\alpha}(x,y) \simeq (xy)^{-\nu-1/2}\left(\frac{xy}{t^{2/\alpha} + (x-y)^2} \wedge 1\right)^{\nu+1/2}$$

$$\times \left[\frac{(1-x)(1-y)}{t^{2/\alpha} + (x-y)^2} \wedge 1\right]\frac{t}{(t^{2/\alpha} + (x-y)^2)^{(\alpha+1)/2}},$$

uniformly in $x, y \in (0, 1)$ and $0 < t \leq T$.

At the end of this section we will complement these results by deriving, directly and for all $\nu > -1$, sharp large time bounds for the kernels $G_t^{\nu,\alpha}(x,y)$, $0 < \alpha \leq 2$.

To prove Theorem 3.3, we need the following.

Lemma 3.5. *Let $d \geq 1$ and $x, y, t, c > 0$ be fixed. Then*

$$\int_{S^{d-1}} \exp\left(-\frac{|\mathbf{x} - y\xi|^2}{ct}\right) d\sigma(\xi)$$

$$= (2\pi)^{\nu+1} \exp\left(-\frac{x^2 + y^2}{ct}\right) \left(\frac{ct}{2xy}\right)^{\nu} I_{\nu}\left(\frac{2xy}{ct}\right),$$

where $\mathbf{x} = (x, 0, \ldots, 0) \in \mathbf{R}^d$, $\nu = d/2 - 1$ and I_{ν} denotes the modified Bessel function of the first kind and order ν.

Proof. We first deal with the case $d \geq 2$. Observe that for $\xi \in S^{d-1}$

$$\exp\left(-\frac{|\mathbf{x} - y\xi|^2}{ct}\right) = \exp\left(-\frac{x^2 + y^2}{ct}\right) \exp\left(\frac{2xy\xi_1}{ct}\right).$$

Thus, the integrand is a zonal function depending only on ξ_1. In this case, the integration over S^{d-1} reduces to a one-dimensional integral against $d\xi_1$, see for instance the proof of [**31,** Chapter IV, Corollary 2.16]. Precisely, we have

$$\int_{S^{d-1}} \exp\left(\frac{2xy\xi_1}{ct}\right) d\sigma(\xi) = \sigma(S^{d-2}) \int_{-1}^{1} \exp\left(\frac{2xy}{ct}\xi_1\right)(1-\xi_1^2)^{(d-3)/2} d\xi_1,$$

where $\sigma(S^{d-2}) = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ is the surface area measure of S^{d-2}. To evaluate the last integral we invoke Schläfli's Poisson type representation for the Bessel function I_{ν} (cf., [**32,** Chapter III, Section 3·71, (9)])

$$I_{\nu}(z) = \frac{1}{\sqrt{\pi}2^{\nu}\Gamma(\nu + 1/2)} z^{\nu} \int_{-1}^{1} \exp(zs)(1 - s^2)^{\nu-1/2} ds, \quad \nu > -1/2.$$

Now putting all the facts together we arrive at the desired identity.

When $d = 1$ the identity is verified directly, taking into account that (cf., [**32,** Chapter III, Section 3·71, (10)]) $I_{-1/2}(z) = (2/(\pi z))^{1/2} \cosh z$. □

Proof of Theorem 3.3. The large time behavior is a consequence of Theorem 2.2 and Theorem 3.1. To show the remaining estimates we

use in addition Lemma 3.5. This produces the bounds

$$(xy)^{-\nu} \left[\frac{(1-x)(1-y)}{t} \wedge 1 \right] \frac{1}{t} \exp\left(-c\frac{x^2+y^2}{t} \right) I_\nu\left(c\frac{2xy}{t} \right)$$

$$\lesssim G_t^{\nu,2}(x,y) \lesssim$$

$$(xy)^{-\nu} \left[\frac{(1-x)(1-y)}{t} \wedge 1 \right] \frac{1}{t} \exp\left(-\frac{x^2+y^2}{ct} \right) I_\nu\left(\frac{2xy}{ct} \right).$$

Now the conclusion follows by applying the standard asymptotics for I_ν, $\nu > -1$,

$$I_\nu(z) \simeq z^\nu, \quad z \to 0^+, \qquad I_\nu(z) \simeq z^{-1/2}\exp(z), \quad z \to \infty,$$

together with the fact that I_ν is continuous on $(0,\infty)$. $\qquad\qquad\square$

In order to prove Theorem 3.4 we first analyze the behavior of a one-dimensional integral depending on several parameters.

Lemma 3.6. *Let γ and η be such that $\gamma > \eta + 1 > 0$. Then, for $0 < B < A < D$,*

$$\int_{-1}^1 \frac{(1-s^2)^\eta \, ds}{(D-Bs)(A-Bs)^\gamma} \simeq \frac{1}{(D-B)A^{\eta+1}(A-B)^{\gamma-\eta-1}}.$$

Proof. Observe that

$$\int_{-1}^1 \frac{(1-s^2)^\eta \, ds}{(D-Bs)(A-Bs)^\gamma} \simeq \int_0^1 \frac{(1-s^2)^\eta \, ds}{(D-Bs)(A-Bs)^\gamma}$$

$$\simeq \int_0^1 \frac{(1-s)^\eta \, ds}{(D-Bs)(A-Bs)^\gamma},$$

so it is enough to analyze the last integral, which we further denote by $\mathcal{I}$. It is convenient to consider the following three cases. Altogether, they cover all admissible configurations of A, B, D.

Case 1. $\mathbf{D > A \geq 2B}$. Since in this case $A - Bs \simeq A \simeq A - B$ and $D - Bs \simeq D \simeq D - B$, for $s \in (0,1)$, the conclusion is trivial.

Case 2. $\mathbf{D \geq 2B > A > B}$. Now $D - Bs \simeq D \simeq D - B$ for $s \in (0,1)$, and therefore

$$\mathcal{I} \simeq \frac{1}{D-B} \int_0^1 \frac{(1-s)^\eta \, ds}{(A-Bs)^\gamma}.$$

Changing the variable $u = 1 - s$ we get

$$\mathcal{I} \simeq \frac{1}{D-B} \int_0^1 \frac{u^\eta \, du}{(A-B+Bu)^\gamma}.$$

Splitting the last integral at the point $u = (A-B)/B$ and estimating the resulting integrals separately we obtain

$$\int_0^{(A-B)/B} \frac{u^\eta \, du}{(A-B+Bu)^\gamma} \simeq \frac{1}{(A-B)^\gamma} \int_0^{(A-B)/B} u^\eta \, du$$

$$\simeq \frac{1}{A^{\eta+1}(A-B)^{\gamma-\eta-1}}$$

and

$$\int_{(A-B)/B}^1 \frac{u^\eta \, du}{(A-B+Bu)^\gamma} \lesssim \frac{1}{B^\gamma} \int_{(A-B)/B}^1 u^{\eta-\gamma} \, du$$

$$\lesssim \frac{1}{B^\gamma} \left(\frac{A-B}{B} \right)^{\eta-\gamma+1}$$

$$\simeq \frac{1}{A^{\eta+1}(A-B)^{\gamma-\eta-1}}.$$

This implies the desired conclusion.

Case 3. $2B > D > A > B$. Changing the variable $u = 1 - s$ we get

$$\mathcal{I} = \int_0^1 \frac{u^\eta \, du}{(D-B+Bu)(A-B+Bu)^\gamma}.$$

We now split the above integral at the points $u_1 = (A-B)/B$ and $u_2 = (D-B)/B$ and estimate the resulting integrals separately. We have

$$\int_0^{(A-B)/B} \frac{u^\eta \, du}{(D-B+Bu)(A-B+Bu)^\gamma}$$

$$\simeq \frac{1}{(D-B)(A-B)^\gamma} \int_0^{(A-B)/B} u^\eta \, du$$

$$\simeq \frac{1}{(D-B)A^{\eta+1}(A-B)^{\gamma-\eta-1}}.$$

Further,

$$\int_{(A-B)/B}^{(D-B)/B} \frac{u^\eta \, du}{(D-B+Bu)(A-B+Bu)^\gamma}$$

$$\lesssim \frac{1}{(D-B)B^\gamma} \int_{(A-B)/B}^{(D-B)/B} u^{\eta-\gamma} \, du$$

$$\lesssim \frac{1}{(D-B)B^\gamma} \left(\frac{A-B}{B}\right)^{\eta-\gamma+1}$$

$$\simeq \frac{1}{(D-B)A^{\eta+1}(A-B)^{\gamma-\eta-1}}.$$

Finally,

$$\int_{(D-B)/B}^{1} \frac{u^\eta \, du}{(D-B+Bu)(A-B+Bu)^\gamma}$$

$$\lesssim \frac{1}{B^{\gamma+1}} \int_{(D-B)/B}^{1} u^{\eta-\gamma-1} \, du$$

$$\lesssim \frac{1}{B^{\gamma+1}} \left(\frac{D-B}{B}\right)^{\eta-\gamma}$$

$$\lesssim \frac{1}{(D-B)A^{\eta+1}(A-B)^{\gamma-\eta-1}}.$$

The conclusion again follows.   □

Proof of Theorem 3.4. We assume $d \geq 2$; the case $d = 1$ is a simplified version of what follows. According to Theorem 2.2 we must integrate the right-hand side in (5) with $\mathbf{x} = (x, 0, \ldots, 0) \in B^d$ and $\mathbf{y}$ replaced by $y\xi$, over S^{d-1} and with respect to $d\sigma(\xi)$. To do that, we first use the relation $z \wedge 1 \simeq z/(z+1)$, $z > 0$, in order to switch to a comparable expression that is more suitable for the integration. This leads to

$$G_t^{\nu,\alpha}(x,y) \simeq \int_{S^{d-1}} \frac{(1-x)(1-y)}{t^{2/\alpha}+|\mathbf{x}-y\xi|^2+(1-x)(1-y)}$$

$$\times \frac{t}{(t^{2/\alpha}+|\mathbf{x}-y\xi|^2)^{(d+\alpha)/2}} \, d\sigma(\xi).$$

Here $|\mathbf{x} - y\xi|^2 = x^2 + y^2 - 2xy\xi_1$ so, in fact, the integrated function is zonal. Thus (see the proof of Lemma 3.5)

$$G_t^{\nu,\alpha}(x,y) \simeq (1-x)(1-y)t \int_{-1}^{1} \frac{1}{(t^{2/\alpha} + x^2 + y^2 - 2xy\xi_1)^{(d+\alpha)/2}}$$
$$\times \frac{(1-\xi_1^2)^{(d-3)/2}\,d\xi_1}{(t^{2/\alpha} + x^2 + y^2 + (1-x)(1-y) - 2xy\xi_1)}.$$

Finally, the last integral is handled by Lemma 3.6 applied with $A = t^{2/\alpha} + x^2 + y^2$, $B = 2xy$, $D = A + (1-x)(1-y)$, $\gamma = (d+\alpha)/2$ and $\eta = (d-3)/2 = \nu - 1/2$. Consequently, we obtain

$$G_t^{\nu,\alpha}(x,y) \simeq \frac{(1-x)(1-y)}{(t^{2/\alpha} + (x-y)^2 + (1-x)(1-y))}$$
$$\times \frac{t}{(t^{2/\alpha} + x^2 + y^2)^{\nu+1/2}(t^{2/\alpha} + (x-y)^2)^{(\alpha+1)/2}}.$$

This, combined with the relations $x^2 + y^2 \simeq (x-y)^2 + xy$ and $z/(1+z) \simeq z \wedge 1$, $z > 0$, finishes the proof. $\square$

We close this section by describing the long time behavior of $G_t^{\nu,\alpha}(x,y)$ for all $\nu > -1$ and $0 < \alpha \le 2$.

Theorem 3.7. *Let $\nu > -1$ and $0 < \alpha \le 2$. There exists $T > 0$ such that*
$$G_t^{\nu,\alpha}(x,y) \simeq (1-x)(1-y)\exp\left(-t\lambda_{1,\nu}^{\alpha}\right),$$
for $x,y \in (0,1)$ and $t \ge T$. Moreover, the upper bound holds for an arbitrary fixed $T > 0$.

Proof. We decompose, see (1),

$$G_t^{\nu,\alpha}(x,y) = e^{-t\lambda_{1,\nu}^{\alpha}}\phi_1^{\nu}(x)\phi_1^{\nu}(y) + \sum_{n=2}^{\infty} e^{-t\lambda_{n,\nu}^{\alpha}}\phi_n^{\nu}(x)\phi_n^{\nu}(y).$$

To treat the first term here we note that

$$\phi_1^{\nu}(x) \simeq (1-x), \quad x \in (0,1).$$

Indeed, $\phi_1^{\nu}(x)$ is strictly positive for $x \in (0,1)$, continuous for $x \in [0,1]$ (see [**15**, (2.3)]), $\phi_1^{\nu}(0) > 0$, $\phi_1^{\nu}(1) = 0$ and $[(d/dx)\phi_1^{\nu}](1) = -\sqrt{2}\lambda_{1,\nu} <$

0. Here the last identity follows from the formula

$$(6) \qquad \frac{d}{dx}\phi_n^\nu(x) = -x^{-\nu-1/2}\lambda_{n,\nu}\, d_{n,\nu}(\lambda_{n,\nu}x)^{1/2}J_{\nu+1}(\lambda_{n,\nu}x),$$

which is a straightforward consequence of the differentiation rule for J_ν, see [**15**, (2.1)].

The proof will be finished once we show that the bound

$$\left|\sum_{n=2}^{\infty} e^{-t\lambda_{n,\nu}^\alpha}\phi_n^\nu(x)\phi_n^\nu(y)\right| \lesssim (1-x)(1-y)e^{-t\lambda_{1,\nu}^\alpha}e^{-t\varepsilon},$$

holds uniformly in $x, y \in (0,1)$ and $t \geq T$, for an arbitrary fixed $T > 0$ and certain $\varepsilon > 0$. To proceed, we first justify the bound

$$(7) \qquad \left|\phi_n^\nu(x)\right| \lesssim (1-x)n^{\nu+2}, \quad x \in (0,1),\ n \geq 1.$$

By the mean value theorem, for each $x \in (0,1)$ there exists $\theta \in (x,1)$ such that

$$\left|\phi_n^\nu(x)\right| = \left|\phi_n^\nu(x) - \phi_n^\nu(1)\right| = (1-x)\left|\frac{d}{dx}\phi_n^\nu(x)\Big|_{x=\theta}\right|.$$

To estimate the last derivative we combine (6) with the bounds given in [**15**, Section 2], see [**15**, (1.3),(2.5),(2.6)], getting

$$\left|\frac{d}{dx}\phi_n^\nu(x)\Big|_{x=\theta}\right| \lesssim n\big(1 \vee n^{\nu+1/2}\big) \leq n^{\nu+2}, \quad n \geq 1,\ \theta \in (0,1).$$

This gives (7). Now we may write

$$\sum_{n=2}^{\infty} e^{-t\lambda_{n,\nu}^\alpha}\left|\phi_n^\nu(x)\phi_n^\nu(y)\right|$$

$$\lesssim (1-x)(1-y)e^{-t\lambda_{1,\nu}^\alpha}e^{-t\varepsilon}\sum_{n=2}^{\infty} n^{2\nu+4}e^{-t(\lambda_{n,\nu}^\alpha - \lambda_{1,\nu}^\alpha - \varepsilon)}.$$

Choosing ε such that $0 < \varepsilon < \lambda_{2,\nu}^\alpha - \lambda_{1,\nu}^\alpha$ and taking into account that $\lambda_{n,\nu} \simeq n$, $n \to \infty$ (cf., [**15**, (2.6)]), we see that the last series is bounded uniformly in $t \geq T$, for any fixed $T > 0$. The conclusion follows. $\qquad\square$

4. Comments and remarks.

4.1. Elementary cases. There are only two cases, $\nu = \pm 1/2$, when the Fourier-Bessel system has an explicit form, i.e., $d_{n,\nu}$, J_ν and $\lambda_{n,\nu}$ are all given explicitly. We have (see [**14**, Section 1]) $\lambda_{n,-1/2} = \pi(n-1/2)$, $\lambda_{n,1/2} = \pi n$, and

$$\phi_n^{-1/2}(x) = \sqrt{2}\cos\left(\pi(n-1/2)x\right),$$
$$\phi_n^{1/2}(x) = x^{-1}\sqrt{2}\sin(\pi n x).$$

Consequently, the Poisson kernel $G_t^{\nu,1}(x,y)$ can be computed when $\nu = \pm 1/2$. Indeed, using basic trigonometric identities and the formulas (cf., [**28**, 5.4.12 (1), (2)])

$$\sum_{j=1}^{\infty} e^{-Aj}\sin(Bj) = \frac{1}{2}\frac{\sin B}{\cosh A - \cos B},$$

$$\sum_{j=1}^{\infty} e^{-Aj}\cos(Bj) = \frac{1}{2}\frac{\sinh A}{\cosh A - \cos B} - \frac{1}{2},$$

we arrive at

$$G_t^{-1/2,1}(x,y) = \frac{\sinh\frac{\pi t}{2}\cos\frac{\pi}{2}(x-y)}{\cosh\pi t - \cos\pi(x-y)} + \frac{\sinh\frac{\pi t}{2}\cos\frac{\pi}{2}(x+y)}{\cosh\pi t - \cos\pi(x+y)},$$

$$G_t^{1/2,1}(x,y) = \frac{1}{2xy}\left[\frac{\sinh\pi t}{\cosh\pi t - \cos\pi(x-y)} - \frac{\sinh\pi t}{\cosh\pi t - \cos\pi(x+y)}\right].$$

Of course, sharp estimates for $G_t^{\pm 1/2,1}(x,y)$ can be obtained directly from these formulas, though it is not immediate.

The heat kernel $G_t^{\nu,2}(x,y)$ is 'computable' as well for $\nu = \pm 1/2$, in the sense that the resulting expressions (in fact series) do not contain oscillations. For the simpler case $\nu = 1/2$ we have

$$G_t^{1/2,2}(x,y) = \frac{1}{xy}\sum_{j\in\mathbf{Z}}\left[\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x-y-2j)^2}{4t}\right)\right.$$
$$\left. - \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{(x+y-2j)^2}{4t}\right)\right].$$

This is essentially a well-known Jacobi type identity. The series above represents the heat kernel in the setting of the system $\{\psi_n^\nu\}$

with $\nu = 1/2$. Note that the corresponding differential operator is $\widetilde{L}_\nu f(x) = x^{\nu+1/2} L_\nu((\cdot)^{-\nu-1/2} f)(x)$, and we have $\widetilde{L}_{1/2} = -\Delta$. The formula in question can be derived by solving the initial-value problem for the classical heat equation in the interval $(0,1)$ with Dirichlet boundary conditions, see [**8**, Chapter 3, Exercises 3.5–3.7]. Here the trick relies on extending a function f prescribing initial values on $(0,1)$ to an odd function on $(-1,1)$ and then considering its periodic extension $\widetilde{f}$ to **R** with period 2. The solution of the heat equation on **R** with initial values prescribed by $\widetilde{f}$, which is given by convolving $\widetilde{f}$ with the Gauss-Weierstrass kernel, also provides the solution of our initial-value problem in the interval $(0,1)$. The convolution can be written in terms of an integral involving the above series, due to the symmetries of $\widetilde{f}$.

The case $\nu = -1/2$ is slightly more involved, but follows in the same spirit. Now $L_\nu = -\Delta$, and one solves the initial-value problem for the heat equation in $(0,1)$ with the Neumann condition at $x = 0$ and the Dirichlet condition at $x = 1$. The relevant extension $\widetilde{f}$ is obtained by extending f to $(-1,1)$ as an even function, then taking extension to $(-1,3)$ that is antisymmetric with respect to $x = 1$ (i.e., satisfies $f(x) = -f(2-x)$ for $x \in (-1,3)$), and finally extending this function from $(-1,3)$ to **R** as periodic with period 4. Solving the heat equation with initial values $\widetilde{f}$ and taking into account the symmetries of $\widetilde{f}$, one concludes that

$$G_t^{-1/2,2}(x,y) = \sum_{j \in \mathbf{Z}} \left[\frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y-4j)^2}{4t}\right) \right.$$
$$+ \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x+y-4j)^2}{4t}\right)$$
$$- \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y-4(j+1/2))^2}{4t}\right)$$
$$\left. - \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x+y-4(j+1/2))^2}{4t}\right) \right].$$

The formulas for $G_t^{\pm 1/2,2}(x,y)$ are suitable for studying short time behavior since for that only the terms corresponding to $j = 0$ (the case of $\nu = -1/2$) or to $j = 0,1$ (the case of $\nu = 1/2$) are essential. On the other hand, they indicate that the related analysis for general $\nu > -1$

is far from being trivial. We remark that the kernels $G_t^{\pm 1/2,2}(x,y)$ can also be expressed by Jacobi's elliptic theta functions θ_2, θ_3 (see [28, page 792] for the definitions), but this representation does not seem to be very useful for our purposes.

4.2. Transference. Next we comment on the transference from the situation of Euclidean balls to the Fourier-Bessel setting on the interval $(0,1)$. Since the semigroups in both settings are related via the mapping φ from Proposition 2.1, the same is true for many fundamental operators expressible through these semigroups. This concerns, for example, maximal operators of the semigroups, fractional integrals (potential operators) and Laplace transform type multipliers in both settings. Moreover, since Lebesgue measure in B^d transforms to the measure $d\mu_\nu$, $\nu = d/2 - 1$, when projecting via φ to $(0,1)$, mapping properties such as L^p-boundedness and weak type estimates of operators defined in the context of Euclidean balls imply the same kind of mapping properties for the corresponding operators in the Fourier-Bessel setting on $(0,1)$. Similar concepts of transference are well known also in other settings of classical orthogonal expansions, for instance, in the direction *Hermite*⟶*Laguerre*, see [21] and [26, Section 5].

4.3. Rough upper bounds. Another issue to discuss are consequences of the rough estimate (2). Transferring it by means of Theorem 2.2 and Lemma 3.5 to the Fourier-Bessel setting on $(0,1)$ we get

$$(8) \qquad 0 < G_t^{\nu,2}(x,y) \leq W_t^{\nu+1/2}(x,y), \quad x,y \in (0,1), \ t > 0,$$

where $\nu = d/2 - 1$, $d \geq 1$, and

$$W_t^\lambda(x,y) = (xy)^{-\lambda+1/2} \frac{1}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_{\lambda-1/2}\left(\frac{xy}{2t}\right)$$

is the heat kernel in the setting of continuous Fourier-Bessel expansions of type $\lambda > -1/2$ on $(0,\infty)$ (the context of the Hankel transform), see [5] or [6]. The bounds (8) are easily explained from the probabilistic point of view. Indeed, $W_t^\lambda(x,y)$ is the transition probability density for the time scaled Bessel process X_{2t} on $(0,\infty)$, see [7, Appendix I, Section 21] (here for $\lambda \in (-1/2, 1/2)$ the endpoint $x = 0$ is assumed to be reflecting), and $G_t^{\lambda-1/2,2}(x,y)$ is the transition probability density for X_{2t} killed upon leaving (through $x = 1$) the interval $(0,1)$, see [22, Section I]. Heuristically, this probabilistic argument works for

all $\nu > -1$ and justifies the claim that the bounds (8) hold for arbitrary $\nu > -1$, with both inequalities being strict. This claim has several interesting consequences. One of them is, see [**27**, Section 6], the fact that the semigroups $\{T_t^{\nu,\alpha}\}_{t>0}$, $\nu > -1$, $0 < \alpha \leq 2$, are submarkovian symmetric diffusion semigroups (to get this for $\alpha < 2$ we use the subordination principle), which are not Markovian. Similar considerations are valid for the corresponding semigroups in the setting of the system $\{\psi_n^\nu\}$, with the restriction $\nu \in \{-1/2\}\cup[1/2,\infty)$, see [**27**, Proposition 6.1].

4.4. Maximal operators. Interesting consequences of (8) concern mapping properties of the maximal operators

$$T_*^{\nu,\alpha}f = \sup_{t>0} \left|T_t^{\nu,\alpha}f\right|.$$

Since $T_*^{\nu,2}$ is controlled by the analogous maximal operator $W_*^{\nu+1/2}$ in the Bessel setting on $(0,\infty)$ related to the measure $d\mu_\nu$, it inherits mapping properties of $W_*^{\nu+1/2}$. The same is true in the context of the system $\{\psi_n^\nu\}$ and the Bessel setting on $(0,\infty)$ related to Lebesgue measure. These observations allow to transmit positive parts of [**6**, Theorem 2.1] to the Fourier-Bessel setting.

Theorem 4.1. *Let $\nu > -1$ be a half-integer, $1 \leq p < \infty$, $\delta \in \mathbf{R}$. Then the maximal operator $T_*^{\nu,2}$, considered on the measure space $((0,1), x^\delta dx)$, has the following mapping properties:*

(a) *$T_*^{\nu,2}$ is of strong type (p,p) if $p > 1$ and $-1 < \delta < (2\nu+2)p-1$;*
(b) *$T_*^{\nu,2}$ is of weak type (p,p) if $-1 < \delta < (2\nu+2)p-1$ or $\delta = 2\nu+1$;*
(c) *$T_*^{\nu,2}$ is of restricted weak type (p,p) if $-1 < \delta \leq (2\nu + 2)p - 1$.*

Moreover, $T_*^{\nu,2}$ is of strong type (∞,∞).

Analogous results, with appropriate adjustments, hold also in the setting of $\{\psi_n^\nu\}$, see [**6**, Remark 3.2]. Boundedness properties with more general weights, and even in tensor product multi-dimensional Fourier-Bessel settings, can be deduced in a similar manner from the corresponding results in [**5**, **9**]; we leave the details to interested readers. The above consequences partially complement and extend previous results on the heat semigroup maximal operator related to

the system $\{\psi_n^\nu\}$ and obtained in [**13, 20**]. Note that, in view of the subordination principle, the above-mentioned mapping properties are inherited by the maximal operators of the subordinated semigroups. Note also that Theorem 4.1 is not optimal in the sense that it does not take into account the boundary behavior at the right endpoint of $(0, 1)$, which in comparison to the continuous Bessel case is improved by the decay of the heat kernel, see Theorem 3.3. Actually, this lack of optimality pertains to many other weighted results obtained so far for various operators in the Fourier-Bessel setting.

We believe that the results of this paper shed some new light on the problems posed in [**4**] in which the heat semigroup comes into play. In particular, in connection with the question [**4,** Question 6], it becomes clear that there must be a pencil phenomenon associated with the heat semigroup maximal operator related to the system $\{\psi_n^\nu\}$, and its nature is the same as in the case of continuous Fourier-Bessel expansions in the Lebesgue measure setting.

4.5. Final conjecture. We close the paper with the following natural conjecture. Proving (or disproving) it is an interesting and important open problem in the theory of Fourier-Bessel expansions.

Conjecture 4.2. *The estimates from Theorems* 3.3 *and* 3.4 *hold for all $\nu > -1$.*

Added in proof. Recently, Conjecture 4.2 has been confirmed by the authors in the two special cases, $\alpha = 2, 1$. See [**24,** Theorem 1.1] and [**25,** Theorem 2.5], respectively.

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