TWO-WEIGHT MIXED NORM ESTIMATES FOR A GENERALIZED SPHERICAL MEAN RADON TRANSFORM ACTING ON RADIAL FUNCTIONS[*]

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Abstract. We investigate a generalized spherical means operator, in other words the generalized spherical mean Radon transform, acting on radial functions. We establish an integral representation of this operator and find precise estimates of the corresponding kernel. As the main result, we prove two-weight mixed norm estimates for the integral operator, with general power weights involved. This leads to weighted Strichartz-type estimates for solutions to certain Cauchy problems for classical Euler–Poisson–Darboux and wave equations with radial initial data.

Key words. spherical Radon transform, spherical mean, radial function, Hankel transform, Legendre function, kernel estimate, mixed norm estimate, two-weight estimate, wave equation, Euler–Poisson–Darboux equation, axially symmetric solution, Strichartz estimate

AMS subject classifications. Primary, 44A12; Secondary, 42B37, 35L15, 35B07, 35L05, 35Q05

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1. Introduction and preliminaries. The spherical mean Radon transform in $\mathbb{R}^n, n \geq 1$, of a suitable function f is given by

$$Mf(x,t) = \int_{S^{n-1}} f(x - ty)\, d\sigma(y), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where $d\sigma$ is the normalized (probabilistic) uniform measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. This operator returns the mean value of f on the sphere centered at x and of radius t.

The spherical means M are of great importance in analysis and have been widely studied due to interest in their own right, their connections with a number of classical initial-value PDE problems, as well as applications in physical/practical problems. The last pertains to thermoacoustic and photoacoustic tomography, among others, where inverse problems for M play a crucial role. A restriction of the action of M to radially symmetric functions in the spatial variable is still of interest for the very same reasons as just mentioned, but in various aspects admits more explicit analysis, and, therefore, in this situation, one is able to obtain sharper or even optimal results.

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It is by no means possible to give here a reasonably complete account of the results on M obtained so far. Thus we limit ourselves to mentioning only a few of them that inspired this work.

The study of L^p estimates for the maximal operator $M^* f = \sup_{t>0} |Mf(\cdot, t)|$ was initiated by Stein [19]. He proved that M^* is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > n/(n-1)$, provided that $n \geq 3$. Later, Bourgain [1] showed that the same holds for $n = 2$. Duoandikoetxea and Vega [7] investigated weighted inequalities for M^* and its dyadic variant. Restricting himself to radial functions, Leckband [10] proved an endpoint result for M^*, namely that it is of restricted weak type (p, p) for $p = n/(n-1)$, $n \geq 2$. Recently, still in the radial case, Duoandikoetxea, Moyua, and Oruetxebarria [5] obtained weighted estimates for M^* that are sharp for power weights.

A generalization of M arises naturally in connection with a Cauchy problem for the classical Euler–Poisson-Darboux (EPD) equation; see, e.g., [22, 3] and references therein. One considers the transformation

$$M^\beta f(x, t) = \mathcal{F}^{-1}\big(m_\beta(t|\cdot|)\mathcal{F}f\big)(x),$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$ and the function defining the multiplier is given by

$$m_\beta(s) = 2^{\beta+n/2-1}\Gamma(\beta+n/2)\frac{J_{\beta+n/2-1}(s)}{s^{\beta+n/2-1}}, \qquad s > 0,$$

with J_ν denoting the (oscillating) Bessel function of the first kind and order ν. Given $t > 0$, the operator $f \mapsto M^\beta f(\cdot, t)$ extends meromorphically to all complex β with poles at $\beta = -n/2, -n/2 - 1, -n/2 - 2, \ldots$. For $\beta = 0$, one recovers the spherical means, i.e., $M^0 = M$. When $\beta > 0$, there is an integral representation (cf. [20, p. 171])

$$M^\beta f(x, t) = \frac{\Gamma(\beta+n/2)}{\pi^{n/2}\Gamma(\beta)} \int_{|y|<1} \big(1 - |y|^2\big)^{\beta-1} f(x - ty)\, dy,$$

the integration being over the unit ball in $\mathbb{R}^n$. Moreover, one can represent M^β in terms of M via an Erdélyi–Kober fractional integral; see [17] for details.

Essentially the same generalization M^β was considered by Stein [19], where he obtained L^p norm estimates for the associated maximal operator. In the same paper, Stein brings to readers' attention an important link between M^β and a Cauchy problem for the classical wave equation. It is noteworthy that any solution to a general initial-value problem for the wave equation can be expressed in terms of $M^0 = M$ and its time derivatives only, at least in odd dimensions $n \geq 3$; see, e.g., [4] and references therein.

The aim of this paper is to study M^β acting on radially symmetric functions. Such a transformation can be viewed as a family of operators $\{\mathcal{M}_t^{\alpha,\beta} : t > 0\}$ (α being a parameter depending on n, to be specified in a moment) acting on profile functions defined on $\mathbb{R}_+$. Then $\mathcal{M}_t^{\alpha,\beta}$ can be expressed in terms of the (modified) Hankel transform $\mathcal{H}_\alpha$. The latter is defined for $\alpha > -1$ and suitable functions f on $\mathbb{R}_+$ by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty f(y)\frac{J_\alpha(xy)}{(xy)^\alpha}\, d\mu_\alpha(y), \qquad x > 0,$$

where $d\mu_\alpha(y) = y^{2\alpha+1}dy$. It is well known that $\mathcal{H}_\alpha$ extends to an isometry on $L^2(\mathbb{R}_+, d\mu_\alpha)$, which satisfies $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$. For $\alpha = n/2 - 1$, the (modified) Hankel transform corresponds to the Fourier transform in $\mathbb{R}^n$ acting on radial functions.

Thus (for suitable f)

$$(1.1) \qquad \mathcal{M}_t^{\alpha,\beta} f = \mathcal{H}_\alpha \big(m_{\alpha,\beta}(t\cdot) \mathcal{H}_\alpha f \big), \qquad t > 0,$$

where the Hankel multipliers are defined by means of the function

$$m_{\alpha,\beta}(s) = 2^{\alpha+\beta} \Gamma(\alpha+\beta+1) \frac{J_{\alpha+\beta}(s)}{s^{\alpha+\beta}}, \qquad s > 0,$$

and α has the form $\alpha = n/2 - 1$, $n \geq 1$. However, from an analytic point of view, there is no reason for restricting ourselves to the discrete set of α. Accordingly, in what follows we allow a continuous range $\alpha > -1$, actually the largest possible so that the Hankel transform is defined on the whole $L^2(\mathbb{R}_+, d\mu_\alpha)$. There is also a deeper motivation for considering general α, and this is related to certain PDE problems involving Bessel operators rather than the standard Laplacian. We shall always require (1.1) to be well defined on $L^2(\mathbb{R}_+, d\mu_\alpha)$. This happens exactly when $\alpha + \beta \geq -1/2$, that is, when $m_{\alpha,\beta}$ is bounded (this fact follows from basic asymptotics for the Bessel function; see below). If this is the case, then $\mathcal{M}_t^{\alpha,\beta}$, $t > 0$, are (uniformly) bounded operators on the L^2 space.

Our principal objective is to prove two-weight mixed norm $L^p - L^q(L_t^r)$ estimates for $\mathcal{M}_t^{\alpha,\beta}$ with possibly large classes of power weights admitted and possibly wide ranges of the parameters involved. This is motivated by the limiting case $r = \infty$ corresponding to the maximal operator $f \mapsto \sup_{t>0} |\mathcal{M}_t^{\alpha,\beta} f|$ and the related investigations in [19, 1, 7, 10, 5]. However, comparing to $1 \leq r < \infty$, the case $r = \infty$ requires a different and in fact more subtle approach, and therefore it will be treated in a separate paper.

For technical reasons, it is much more convenient to work with an integral operator $M_t^{\alpha,\beta}$ that agrees with $\mathcal{M}_t^{\alpha,\beta}$ in $L^2(\mathbb{R}_+, d\mu_\alpha)$. Thus our strategy is to switch to $M_t^{\alpha,\beta}$ and then find precise estimates of the associated integral kernel $K_t^{\alpha,\beta}(x,z)$ in order to enable a direct and explicit analysis of the operator. The latter relies on estimating first the norm of the kernel in power-weighted $L^r(dt)$, $1 \leq r < \infty$, and then showing two-weight mixed norm estimates for the resulting integral operator independent of the "time" variable t.

As illustrative applications of the mixed norm inequalities obtained, we derive weighted Strichartz-type estimates for solutions of certain initial-value problems for the EPD and wave equations, as well as similar differential problems based on the one-dimensional Bessel operator $L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x} \frac{d}{dx}$. Such results seem to be desirable from the PDE theory perspective.

An interesting aspect of our research is examining the behavior of the kernel $K_t^{\alpha,\beta}(x,z)$. Perhaps a bit surprisingly, there are half-lines and segments in the (α, β) plane where a kind of phase shift occurs. More precisely, the behavior of the kernel is essentially different when (α, β) belongs to those singular sets, compared to the behavior in neighborhoods of those sets. This phenomenon makes statements of the kernel estimates somewhat complicated. Actually, something similar happens also in the case of certain asymptotics for the Legendre functions through which we express the kernel. The literature seems to tacitly omit those "singular" asymptotics. This led us to derive them by ourselves, by means of combining various known facts and some computations. Another topic that seems not to be covered properly by the (at least standard) literature are zeros of Legendre functions. Here we also had to work a bit by ourselves to derive what was needed for the purposes of this paper.

1.1. Integral representation of the radial spherical means $\mathcal{M}_t^{\alpha,\beta}$. It turns out that $\mathcal{M}_t^{\alpha,\beta}$ can be represented as a standard integral operator provided that $\alpha + \beta > -1/2$. In case $\alpha + \beta = -1/2$, there is a singular integral representation, which is much more subtle and not treated in this paper.

Define the kernel

$$K_t^{\alpha,\beta}(x,z) = 2^{\alpha+\beta}\Gamma(\alpha+\beta+1)\int_0^\infty \frac{J_{\alpha+\beta}(ty)}{(ty)^{\alpha+\beta}}\frac{J_\alpha(xy)}{(xy)^\alpha}\frac{J_\alpha(zy)}{(zy)^\alpha}d\mu_\alpha(y).$$

As we shall see, $K_t^{\alpha,\beta}(x,z)$ is well defined for $\alpha > -1$, $\alpha + \beta > -1/2$, $t > 0$, and (in general) $t \neq |x - z|$, $t \neq x + z$. The integral here converges absolutely when $\alpha + \beta > 1/2$; otherwise, the convergence at ∞ is only conditional in the Riemann sense. Notice that the kernel is homogeneous in the sense that

$$(1.2) \qquad K_t^{\alpha,\beta}(x,z) = \frac{1}{s^{2\alpha+2}}K_{t/s}^{\alpha,\beta}\left(\frac{x}{s},\frac{z}{s}\right), \qquad s,t,x,z > 0.$$

This property is of importance from the point of view of analysis related to the operator we now define.

For each $t > 0$, consider the integral operator

$$M_t^{\alpha,\beta}f(x) = \int_0^\infty K_t^{\alpha,\beta}(x,z)f(z)\,d\mu_\alpha(z)$$

on its natural domain $\mathrm{Dom}\,M_t^{\alpha,\beta}$ consisting of all f for which the above integral converges absolutely for a.a. $x > 0$. The following result gives a preliminary link between $\mathcal{M}_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$.

PROPOSITION 1.1. *Let $\alpha > -1$ and $\alpha + \beta > -1/2$. Then, for each $t > 0$, $C_c^\infty(0,\infty) \subset \mathrm{Dom}\,M_t^{\alpha,\beta}$ and*

$$\mathcal{M}_t^{\alpha,\beta}f = M_t^{\alpha,\beta}f, \qquad f \in C_c^\infty(0,\infty).$$

Proof. We shall use the basic asymptotics for the Bessel function,

$$J_\alpha(u) \simeq u^\alpha, \quad u \to 0^+, \qquad J_\alpha(u) = \mathcal{O}(u^{-1/2}), \quad u \to \infty,$$

together with the following fact: if $g = \mathcal{H}_\alpha f$ for some $f \in C_c^\infty(0,\infty)$, then g is continuous, $g(u) = \mathcal{O}(1)$ as $u \to 0^+$, and, for any fixed k, $g(u) = \mathcal{O}(u^{-k})$ as $u \to \infty$; see [13, section 2.1].

Let $f \in C_c^\infty(0,\infty)$. Since, in view of what was said above, $m_{\alpha,\beta}(t\cdot)\mathcal{H}_\alpha f \in L^1(d\mu_\alpha)$ and also $f \in L^1(d\mu_\alpha)$, we can write

$$\frac{\mathcal{M}_t^{\alpha,\beta}f(x)}{2^{\alpha+\beta}\Gamma(\alpha+\beta+1)} = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha}\frac{J_{\alpha+\beta}(ty)}{(ty)^{\alpha+\beta}}\mathcal{H}_\alpha f(y)\,d\mu_\alpha(y)$$

$$(1.3) \qquad = \int_0^\infty\int_0^\infty \frac{J_{\alpha+\beta}(ty)}{(ty)^{\alpha+\beta}}\frac{J_\alpha(xy)}{(xy)^\alpha}\frac{J_\alpha(zy)}{(zy)^\alpha}f(z)\,d\mu_\alpha(z)\,d\mu_\alpha(y), \quad t,x > 0.$$

The proof will be finished once we show that changing the order of integration in (1.3) is legitimate. It is easily seen that this is indeed the case when $\alpha + \beta > 1/2$, since then the last double integral converges absolutely, and one can use Fubini's theorem. However, in case $\alpha + \beta \leq 1/2$, the situation is more delicate, because the integral defining $K_t^{\alpha,\beta}(x,z)$ converges in the Riemann sense, but not absolutely.

To proceed, we assume that $t, x > 0$ are fixed, and f is also fixed and its support is contained in an interval $[A, B]$ with $0 < A < B < \infty$. Splitting the outer integral in (1.3) we reduce the problem to switching the order integration in

$$\int_1^\infty \int_A^B \frac{J_{\alpha+\beta}(ty)}{(ty)^{\alpha+\beta}} \frac{J_\alpha(xy)}{(xy)^\alpha} \frac{J_\alpha(zy)}{(zy)^\alpha} f(z) \, d\mu_\alpha(z) \, d\mu_\alpha(y).$$

Next, we expand each of the three Bessel functions according to the large argument asymptotics (see [21, Chapter VII, section 7·21(1)]),

$$J_\nu(w) = \sqrt{\frac{2}{\pi w}} \cos\left(w - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) + \mathcal{O}(w^{-3/2}),$$

valid for positive w separated from 0. This leads to a splitting of the integrand into eight terms. All the resulting double integrals converge absolutely, except for one, written here up to some factors that can be neglected in further analysis:

$$\int_1^\infty \int_A^B \frac{\cos(ty - c_1) \cos(xy - c_2) \cos(zy - c_2)}{y^{\alpha+\beta+1/2}} f(z) \, dz \, dy.$$

In the last expression we can write the outer integral as a limit of integrals over bounded intervals and exchange the order of integration due to absolute integrability. This means that our task reduces to checking that one can pass within the limit under the integral sign in

$$\lim_{N\to\infty} \int_A^B G_N(z) f(z) \, dz,$$

where

$$G_N(z) = \int_1^N \frac{\cos(ty - c_1) \cos(xy - c_2) \cos(zy - c_2)}{y^{\alpha+\beta+1/2}} \, dy.$$

If we now show that the sequence $\{|G_N|\}$ is controlled by an integrable function over $[A, B]$, then the desired conclusion will follow from the dominated convergence theorem.

To continue, we invoke the product-to-sum formula

$$\cos\theta_1 \cos\theta_2 \cos\theta_3 = \frac{1}{8} \sum_{(e_1,e_2,e_3)\in\{-1,1\}^3} \cos(e_1\theta_1 + e_2\theta_2 + e_3\theta_3),$$

getting

$$4\cos(ty - c_1) \cos(xy - c_2) \cos(zy - c_2)$$
$$= \cos\big(y(t + x + z) - c_1 - 2c_2\big) + \cos\big(y(t + x - z) - c_1\big)$$
$$+ \cos\big(y(t - x + z) - c_1\big) + \cos\big(y(x + z - t) + c_1 - 2c_2\big).$$

We now see that it is enough to verify the existence of an integrable over the $[A, B]$ majorant of $\{|H_N(z)|\}$, where the new sequence is of the form

$$H_N(z) = \int_1^N \frac{\cos\big(y(z - D) + C\big)}{y^\lambda} \, dy,$$

with $0 < \lambda \leq 1$ fixed and $C, D \in \mathbb{R}$ also fixed (of course, it may happen that $D \in [A, B]$).

We treat here a simplified model situation, which gets to the heart of the matter. The general case then requires some elementary technical adjustments, which are left to the reader. Let $A = C = D = 0$. Then, changing the variable of integration, we get

$$H_N(z) = \int_1^N \frac{\cos(yz)}{y^\lambda}\, dy = z^{\lambda-1} \int_z^{Nz} \frac{\cos s}{s^\lambda}\, ds.$$

If $\lambda < 1$, the last integral stays bounded when N and z vary, since $s^{-\lambda}\cos s$ is integrable over $(0,\infty)$ (at ∞ in the Riemann sense only). Thus we see that the required majorant is $H(z) = cz^{\lambda-1}$. When $\lambda = 1$ we split the last integral with respect to the point $1 \wedge (Nz)$ and then easily see that in this case the majorant is $H(z) = c(1 + \log^+ \frac{1}{z})$. □

Later, in section 4, we will see that for each $t > 0$, $L^2(\mathbb{R}_+, d\mu_\alpha) \subset \text{Dom } M_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}_+, d\mu_\alpha)$. This together with Proposition 1.1 implies that $\mathcal{M}_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$ coincide as operators acting on $L^2(\mathbb{R}_+, d\mu_\alpha)$.

1.2. Structure of the paper and notation. The rest of the paper is organized as follows. In section 2, to taste the flavor of the problem and gain a better intuition, we find sharp estimates of the integral kernel of $M_t^{\alpha,\beta}$ in the uncomplicated case when $\alpha > -1/2$ and $\beta > 0$; see Theorem 2.3. This is done by employing a relatively simple positive integral representation for the triple Bessel function integral entering the kernel. Then, with the aid of sharp bounds for certain elementary integrals (see Lemmas 2.1 and 2.2), the result follows in a rather straightforward manner. We also look at a few special cases of the parameters α, β in which the kernel $K_t^{\alpha,\beta}(x,z)$ is totally computable. This reveals, in particular, that one has to be careful when it comes to values of the kernel related to the singular surfaces $t = |x - z|$ and $t = x + z$, even if those values are finite. In section 3, we estimate the kernel $K_t^{\alpha,\beta}(x,z)$ in the general case when $\alpha > -1$ and $\beta > -\alpha - 1/2$. Here the strategy is to express the triple Bessel function integral via suitable Legendre functions and then estimate the resulting expressions by means of Legendre function asymptotics. Since the latter seem to be incomplete, at least in the standard literature, we derive the missing cases ourselves using known facts and formulas and explicit computations. Another important issue we study in this section is the presence or lack of zeros of the Legendre functions, since in the latter case the estimates of the kernel we get are in fact sharp. Our main result on the behavior of $K_t^{\alpha,\beta}(x,z)$ is stated in Theorem 3.3. In section 4, we prove that for each $t > 0$ the integral operator $M_t^{\alpha,\beta}$ is well defined and bounded on $L^2(\mathbb{R}_+, d\mu_\alpha)$. Consequently, the L^2-coincidence between $\mathcal{M}_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$ is established; see Corollary 4.2. In section 5, we estimate the norm of the kernel $K_t^{\alpha,\beta}(x,z)$ in power-weighted $L^r(dt)$. The bounds we get are fairly precise in general and sharp in many cases; see Theorem 5.1. In section 6, we state the main result of the paper, that is, the two-weight mixed norm estimate for $M_t^{\alpha,\beta}$ which is contained in Theorem 6.5. This is preceded by a sharp analysis of an auxiliary integral operator emerging from the precise absolute estimates for the kernel $K_t^{\alpha,\beta}(x,z)$ obtained previously in Theorem 3.3. Finally, section 7 is devoted to applications of the mixed norm estimates. These pertain to weighted Strichartz-type estimates for solutions to certain radial initial-value problems for classical EPD and wave equations, as well as Bessel operator–based counterparts of these equations.

Throughout the paper we use fairly standard notation. Thus $\mathbb{R}_+ = (0,\infty)$. The symbols "$\vee$" and "$\wedge$" mean the operations of taking maximum and minimum, respectively. We write $X \lesssim Y$ to indicate that $X \leq CY$ with a positive constant C

independent of significant quantities. We shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$.

For the sake of brevity, we shall omit $\mathbb{R}_+$ when denoting L^p spaces related to the measure space $(\mathbb{R}_+, d\mu_\alpha)$. Given a nonnegative weight w, we denote by $L^p(w^p d\mu_\alpha)$ the weighted L^p space with respect to the measure μ_α. This means that $f \in L^p(w^p d\mu_\alpha)$ if and only if $wf \in L^p(d\mu_\alpha)$. By convention, $L^\infty(w^\infty d\mu_\alpha)$ consists of all measurable functions f such that wf is essentially bounded on $\mathbb{R}_+$, and the norm of f in that space is $\|wf\|_\infty$. We write $L^p_{\mathrm{rad}}(\ldots)$ for the subspace of $L^p(\ldots)$ consisting of radial functions. As usual, for $1 \leq p \leq \infty$, p' denotes its conjugate exponent, $1/p + 1/p' = 1$.

2. Pointwise kernel estimates I: A special case. In this section we prove, by elementary methods, sharp estimates of the kernel $K_t^{\alpha,\beta}(x,z)$ in case $\alpha > -1/2$ and $\beta > 0$. For such parameters the kernel is given by means of a well-studied generalization of the Weber–Schafheitlin integral. More precisely, formula [21, Chapter XIII, section 13·46(1)] implies, for $t, x, z > 0$ such that $t \neq |x - z|$ and $t \neq x + z$, that

$$(2.1) \qquad K_t^{\alpha,\beta}(x,z) = c_{\alpha,\beta}\, t^{-2(\alpha+\beta)} \int_0^A \left(t^2 - x^2 - z^2 + 2xz \cos\theta\right)^{\beta-1} \sin^{2\alpha}\theta\, d\theta.$$

Here $c_{\alpha,\beta} = 2\Gamma(\alpha + \beta + 1)/(\sqrt{\pi}\Gamma(\alpha + 1/2)\Gamma(\beta))$ and

$$A = \begin{cases} 0, & t < |x - z|, \\ \arccos\left(\frac{x^2+z^2-t^2}{2xz}\right), & |x - z| < t < x + z, \\ \pi, & t > x + z. \end{cases}$$

Notice that for α and β under consideration, $K_t^{\alpha,\beta}(x,z)$ is nonnegative, $K_t^{\alpha,\beta}(x,z) = 0$ if $t < |x - z|$, and $K_t^{\alpha,\beta}(x,z) > 0$ when $t > |x - z|$.

2.1. Two simple technical results. We need precise estimates of the following integrals:

$$I_{\alpha,\gamma}(B) := \int_{-1}^1 (1 - Bs)^\gamma (1 - s^2)^{\alpha-1/2}\, ds, \qquad 0 \leq B \leq 1,$$

$$J_{\alpha,\beta,\gamma}(D) := \int_0^1 (D - s)^{\alpha-1/2}(1 - s)^{\beta-1} s^\gamma\, ds, \qquad D \geq 1.$$

LEMMA 2.1. *Let $\alpha > -1/2$ and $\gamma \in \mathbb{R}$ be fixed. Then*

$$I_{\alpha,\gamma}(B) \simeq \begin{cases} (1 - B)^{\alpha+\gamma+1/2}, & \alpha + \gamma + 1/2 < 0, \\ 1 + \log\frac{1}{1-B}, & \alpha + \gamma + 1/2 = 0, \\ 1, & \alpha + \gamma + 1/2 > 0 \end{cases}$$

uniformly in $0 \leq B \leq 1$.

The case $B = 1$ in the statement of Lemma 2.1 and in the proof below should be understood in the usual limiting sense.

Proof of Lemma 2.1. Assume, to begin, that $\gamma \geq 0$. Then the essential contribution to $I_{\alpha,\gamma}(B)$ comes from integration between -1 and 0. Therefore, taking into account that $1 - Bs \simeq 1$ and $1 - s \simeq 1$ when $-1 < s < 0$, we can write

$$I_{\alpha,\gamma}(B) \simeq \int_{-1}^0 (1 - Bs)^\gamma (1 - s^2)^{\alpha-1/2}\, ds \simeq \int_{-1}^0 (1 + s)^{\alpha-1/2}\, ds \simeq 1.$$

This agrees with the asserted estimate, since $\gamma \geq 0$ implies $\alpha + \gamma + 1/2 > 0$.

Assume next that $\gamma < 0$. Now the essential contribution to $I_{\alpha,\gamma}(B)$ comes from integration between 0 and 1, and we have

$$(2.2) \qquad I_{\alpha,\gamma}(B) \simeq \int_0^1 (1 - Bs)^\gamma (1 - s)^{\alpha - 1/2} \, ds.$$

When $\alpha + \gamma + 1/2 > 0$, we use the straightforward bounds

$$\int_0^1 (1 - s)^{\alpha - 1/2} \, ds \lesssim I_{\alpha,\gamma}(B) \lesssim \int_0^1 (1 - s)^{\alpha + \gamma - 1/2} \, ds$$

to conclude that $I_{\alpha,\gamma}(B) \simeq 1$. Thus it remains to treat the case $\alpha + \gamma + 1/2 \leq 0$. Here we may assume that $B > 1/2$, since otherwise $I_{\alpha,\gamma}(B) \simeq 1$, as needed. Changing the variable of integration $s = 1 - \frac{1-B}{B} w$ in (2.2) and remembering that now $B \simeq 1$, we get

$$I_{\alpha,\gamma}(B) \simeq (1 - B)^{\alpha + \gamma + 1/2} \int_0^{B/(1-B)} (1 + w)^\gamma w^{\alpha - 1/2} \, dw.$$

Denoting by $\widetilde{I}_{\alpha,\gamma}(B)$ the last integral, we see that

$$\widetilde{I}_{\alpha,\gamma}(B) \simeq 1 + \int_1^{B/(1-B)} w^{\alpha + \gamma - 1/2} \, dw \simeq \begin{cases} 1, & \alpha + \gamma + 1/2 < 0, \\ 1 + \log \frac{B}{1-B}, & \alpha + \gamma + 1/2 = 0, \end{cases}$$

where in the log case $\frac{B}{1-B}$ may be replaced by $\frac{1}{1-B}$. The conclusion follows. □

LEMMA 2.2. *Let $\alpha \in \mathbb{R}$, $\beta > 0$, and $\gamma > -1$ be fixed. Then*

$$J_{\alpha,\beta,\gamma}(D) \simeq D^{\alpha - 1/2}, \qquad D \geq 2,$$

and

$$J_{\alpha,\beta,\gamma}(D) \simeq \begin{cases} (D - 1)^{\alpha + \beta - 1/2}, & \alpha + \beta < 1/2, \\ 1 + \log \frac{1}{D-1}, & \alpha + \beta = 1/2, \\ 1, & \alpha + \beta > 1/2 \end{cases}$$

uniformly in $1 \leq D < 2$.

The case $D = 1$ in the statement of Lemma 2.2 and in its proof is understood in the limiting sense.

Proof of Lemma 2.2. Split the integral defining $J_{\alpha,\beta,\gamma}(D)$ according to the intervals $(0, 1/2)$ and $(1/2, 1)$, and denote the resulting integrals by $J^0_{\alpha,\beta,\gamma}(D)$ and $J^1_{\alpha,\beta,\gamma}(D)$, respectively. Then we have

$$J^0_{\alpha,\beta,\gamma}(D) \simeq D^{\alpha - 1/2} \int_0^{1/2} s^\gamma \, ds \simeq D^{\alpha - 1/2}.$$

For the complementary integral we write

$$J^1_{\alpha,\beta,\gamma}(D) \simeq \int_{1/2}^1 (D - s)^{\alpha - 1/2} (1 - s)^{\beta - 1} \, ds.$$

Changing now the variable of integration $s = 1 - (D - 1)w$ leads to

$$J^1_{\alpha,\beta,\gamma}(D) \simeq (D - 1)^{\alpha + \beta - 1/2} \int_0^{\frac{1}{2(D-1)}} (1 + w)^{\alpha - 1/2} w^{\beta - 1} \, dw.$$

For $D \geq 3/2$, it follows that

$$J^1_{\alpha,\beta,\gamma}(D) \simeq (D-1)^{\alpha+\beta-1/2} \int_0^{\frac{1}{2(D-1)}} w^{\beta-1}\, dw \simeq (D-1)^{\alpha-1/2} \simeq D^{\alpha-1/2}.$$

On the other hand, if $1 \leq D < 3/2$, then

$$J^1_{\alpha,\beta,\gamma}(D) \simeq (D-1)^{\alpha+\beta-1/2} \left[1 + \int_1^{\frac{1}{2(D-1)}} w^{\alpha+\beta-3/2}\, dw \right]$$

$$\simeq \begin{cases} (D-1)^{\alpha+\beta-1/2}, & \alpha+\beta < 1/2, \\ 1 + \log\frac{1}{D-1}, & \alpha+\beta = 1/2, \\ 1, & \alpha+\beta > 1/2. \end{cases}$$

Combining the above estimates of $J^0_{\alpha,\beta,\gamma}(D)$ and $J^1_{\alpha,\beta,\gamma}(D)$ we arrive at the desired conclusion. $\qquad\Box$

2.2. Estimates of the kernel $K_t^{\alpha,\beta}(x,z)$. We are now ready to prove sharp estimates of $K_t^{\alpha,\beta}(x,z)$. Recall that the kernel vanishes in the region $\{t, x, z > 0 : t < |x-z|\}$.

THEOREM 2.3. *Let $\alpha > -1/2$ and $\beta > 0$ be fixed. Let $t, x, z > 0$. Then*

$$K_t^{\alpha,\beta}(x,z) \simeq \frac{(xz)^{-\alpha-1/2}}{t^{2\alpha+2\beta}} \left[t^2 - (x-z)^2 \right]^{\alpha+\beta-1/2} \begin{cases} \left(\frac{(x+z)^2-t^2}{xz} \right)^{\alpha+\beta-1/2}, & \alpha+\beta < 1/2, \\ 1 + \log\left(\frac{xz}{(x+z)^2-t^2} \right), & \alpha+\beta = 1/2, \\ 1, & \alpha+\beta > 1/2 \end{cases}$$

uniformly in $|x-z| < t < x+z$, and

$$K_t^{\alpha,\beta}(x,z) \simeq \frac{1}{t^{2\alpha+2\beta}} \left[t^2 - (x-z)^2 \right]^{\beta-1} \begin{cases} \left(\frac{t^2-(x+z)^2}{t^2-(x-z)^2} \right)^{\alpha+\beta-1/2}, & \alpha+\beta < 1/2, \\ 1 + \log\left(\frac{t^2-(x-z)^2}{t^2-(x+z)^2} \right), & \alpha+\beta = 1/2, \\ 1, & \alpha+\beta > 1/2 \end{cases}$$

uniformly in $t > x+z$.

Taking into account the relation

$$(2.3) \qquad xz \simeq \left[t^2 - (x-z)^2 \right] \vee \left[(x+z)^2 - t^2 \right], \qquad |x-z| < t < x+z,$$

we see that the behavior of $K_t^{\alpha,\beta}(x,z)$ depends on x and z only through $|x-z|$ and $x+z$. Moreover, this behavior depends essentially on the distances from t^2 to $(x-z)^2$ and $(x+z)^2$, and some singularities occur when any of them tends to zero. It is perhaps interesting to observe that the bounds from Theorem 2.3 can be written in a more compact way as

$$K_t^{\alpha,\beta}(x,z) \simeq t^{-2(\alpha+\beta)} \left[\left(t^2 - (x-z)^2 \right) \vee \left| t^2 - (x+z)^2 \right| \right]^{-\alpha-1/2}$$

$$\times \begin{cases} \left[\left(t^2 - (x-z)^2 \right) \wedge \left| t^2 - (x+z)^2 \right| \right]^{\alpha+\beta-1/2}, & \alpha+\beta < 1/2, \\ 1 + \log\frac{t^2-(x-z)^2}{[t^2-(x-z)^2]\wedge|t^2-(x+z)^2|}, & \alpha+\beta = 1/2, \\ \left(t^2 - (x-z)^2 \right)^{\alpha+\beta-1/2}, & \alpha+\beta > 1/2 \end{cases}$$

uniformly in $t > |x - z|$ such that $t \neq x + z$.

Proof of Theorem 2.3. We distinguish three cases emerging from splitting the range of t^2 according to the points $(x - z)^2$, $x^2 + z^2$, and $(x + z)^2$. Observe that the middle point is the geometric center of the interval defined by the other points as endpoints.

Case 1. $t^2 > (x + z)^2$. Changing the variable of integration $\cos\theta = -s$ in (2.1), we get

$$K_t^{\alpha,\beta}(x, z) = c_{\alpha,\beta}\, t^{-2(\alpha+\beta)} \int_{-1}^{1} \left(t^2 - x^2 - z^2 - 2xzs\right)^{\beta-1}(1 - s^2)^{\alpha-1/2}\, ds$$

$$= c_{\alpha,\beta}\, t^{-2(\alpha+\beta)}\left(t^2 - x^2 - z^2\right)^{\beta-1} I_{\alpha,\beta-1}\left(\frac{2xz}{t^2 - x^2 - z^2}\right).$$

Now an application of Lemma 2.1 gives

$$K_t^{\alpha,\beta}(x, z) \simeq t^{-2(\alpha+\beta)}\left(t^2 - x^2 - z^2\right)^{\beta-1} \begin{cases} \left(\frac{t^2-(x+z)^2}{t^2-x^2-z^2}\right)^{\alpha+\beta-1/2}, & \alpha + \beta < 1/2, \\ 1 + \log\frac{t^2-x^2-z^2}{t^2-(x+z)^2}, & \alpha + \beta = 1/2, \\ 1, & \alpha + \beta > 1/2. \end{cases}$$

Since $t^2 - x^2 - z^2 \simeq t^2 - (x - z)^2$, this is equivalent to the bounds of the theorem.

Case 2. $x^2 + z^2 < t^2 < (x + z)^2$. Changing the variable of integration $\cos\theta = 1 - \frac{t^2-(x-z)^2}{2xz}s$ in (2.1) and then simplifying the resulting expression, we see that

(2.4)
$$K_t^{\alpha,\beta}(x, z) = c_{\alpha,\beta}\, t^{-2(\alpha+\beta)}(2xz)^{-2\alpha}\left[t^2 - (x - z)^2\right]^{2\alpha+\beta-1} J_{\alpha,\beta,\alpha-1/2}\left(\frac{4xz}{t^2 - (x - z)^2}\right).$$

Since $t^2 > x^2 + z^2$ is equivalent to $\frac{4xz}{t^2-(x-z)^2} < 2$, Lemma 2.2 implies

$$K_t^{\alpha,\beta}(x, z)$$

$$\simeq t^{-2(\alpha+\beta)}(xz)^{-2\alpha}\left[t^2 - (x - z)^2\right]^{2\alpha+\beta-1} \begin{cases} \left(\frac{(x+z)^2-t^2}{t^2-(x-z)^2}\right)^{\alpha+\beta-1/2}, & \alpha + \beta < 1/2, \\ 1 + \log\frac{t^2-(x-z)^2}{(x+z)^2-t^2}, & \alpha + \beta = 1/2, \\ 1, & \alpha + \beta > 1/2, \end{cases}$$

which with the aid of (2.3) leads to the estimates of the theorem.

Case 3. $(x - z)^2 < t^2 \leq x^2 + z^2$. In view of (2.3), the estimate in question can be stated simply as

$$K_t^{\alpha,\beta}(x, z) \simeq t^{-2(\alpha+\beta)}(xz)^{-\alpha-1/2}\left[t^2 - (x - z)^2\right]^{\alpha+\beta-1/2}.$$

But this is a straightforward consequence of (2.4) and Lemma 2.2, since $t^2 \leq x^2 + z^2$ means that $\frac{4xz}{t^2-(x-z)^2} \geq 2$.

The proof of Theorem 2.3 is complete. □

2.3. Some special elementary cases. It is interesting to observe that for $\beta = 0$ the kernel $K_t^{\alpha,\beta}(x, z)$ can be computed explicitly. To this end let $\alpha > -1/2$ and $\beta = 0$. We have

$$K_t^{\alpha,0}(x, z) = 2^\alpha \Gamma(\alpha + 1)(txz)^{-\alpha} \int_0^\infty J_\alpha(ty)J_\alpha(xy)J_\alpha(zy)\, y^{1-\alpha}\, dy.$$

By virtue of formula [21, Chapter XIII, section 13·46(3)] (see also [16, formula (14) on p. 230]), the kernel vanishes if either $t < |x - z|$ or $t > x + z$, and for $|x - z| < t < x + z$ we have

$$(2.5) \quad K_t^{\alpha,0}(x, z) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} 2^{2\alpha-1} \Gamma(\alpha + 1/2)} (txz)^{-2\alpha} \left(\left[t^2 - (x - z)^2 \right] \left[(x + z)^2 - t^2 \right] \right)^{\alpha - 1/2}.$$

In [10, 5], standard spherical means of radial functions in $\mathbb{R}^n$, $n \geq 2$, were considered. These means are represented via the one-dimensional kernel

$$(2.6) \qquad L_t^n(x, z) = \frac{4\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{\pi}} \left[\frac{2(b^2 - t^2)^{1/2}(t^2 - a^2)^{1/2}}{b^2 - a^2} \right]^{n-3} \frac{t^{2-n}}{b^2 - a^2},$$

where $a = |x - z|$, $b = x + z$, $|x - z| < t < x + z$; the related measure of integration is $z^{n-1} dz$. One can generalize (2.6) by letting $n = 2\alpha + 2$ and considering a continuous range $\alpha > -1/2$. Then it is straightforward to check that

$$L_t^{2\alpha+2}(x, z) = K_t^{\alpha,0}(x, z), \qquad |x - z| < t < x + z.$$

Moreover, the measure $z^{n-1} dz$ becomes $d\mu_\alpha(z)$. Thus the kernel $K_t^{\alpha,\beta}(x, z)$ generalizes $L_t^{2\alpha+2}(x, z)$.

We now look closer at the more elementary case $\beta = 0$, $\alpha = 1/2$, to see that one indeed has to be careful when dealing with values of $K_t^{\alpha,\beta}(x, z)$ in the singular cases $t = |x - z|$ and $t = x + z$ (anyway, values of the kernel for those t, x, z are irrelevant for our purposes). We have

$$K_t^{1/2,0}(x, z) = \frac{2}{\pi} (txz)^{-1} \int_0^\infty \sin(ty) \sin(xy) \sin(zy) \frac{dy}{y}.$$

Using [15, formula (3) on p. 411], we find that

$$K_t^{1/2,0}(x, z) = \frac{1}{2} (txz)^{-1} \begin{cases} 0, & |x - z| > t \text{ or } t > x + z, \\ 1/2, & t = |x - z| \text{ or } t = x + z, \\ 1, & |x - z| < t < x + z. \end{cases}$$

Another example in the same spirit is the case $\alpha = -1/2$, $\beta = 1$, computed by means of [15, formula (21) on pp. 406–407]:

$$K_t^{-1/2,1}(x, z) = \frac{1}{t} \begin{cases} 0, & t < |x - z|, \\ 1/4, & t = |x - z|, \\ 1/2, & |x - z| < t < x + z, \\ 3/4, & t = x + z, \\ 1, & t > x + z. \end{cases}$$

The careful reader probably has noticed that the explicit formulas just given are not consistent with the estimates of Theorem 2.3. More precisely, a kind of phase shift occurs for $\alpha \geq 1/2$ and $\beta = 0$, as well as for $\alpha = -1/2$ and $\beta = 1$. This interesting and perhaps a bit unexpected phenomenon will be fully revealed in Theorem 3.3.

Further explicit formulas for the kernel are implicitly contained in section 3.2 below; see Remark 3.1.

3. Pointwise kernel estimates II: General case. Recall that the kernel we are dealing with is

$$K_t^{\alpha,\beta}(x,z) = \frac{2^{\alpha+\beta}\Gamma(\alpha+\beta+1)}{t^{\alpha+\beta}(xz)^\alpha} \int_0^\infty J_{\alpha+\beta}(ty)J_\alpha(xy)J_\alpha(zy)y^{1-\alpha-\beta}\, dy, \quad t,x,z > 0.$$

In the previous section we found sharp estimates of this kernel when $\alpha > -1/2$ and $\beta > 0$. Now we consider all $\alpha > -1$ and $\beta > -\alpha - 1/2$, that is, all (α,β) for which the kernel is defined. Our aim is to find possibly precise estimates of $K_t^{\alpha,\beta}(x,z)$. Note that one cannot hope for sharpness for all α and β in question, since in general the kernel takes both positive and negative values.

Denote the integral entering the kernel by $\mathcal{I}_{\alpha,\beta}$,

$$(3.1) \qquad \mathcal{I}_{\alpha,\beta} = \frac{1}{2^{\alpha+\beta}\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}(xz)^\alpha K_t^{\alpha,\beta}(x,z).$$

In order to estimate the kernel, we shall first study $\mathcal{I}_{\alpha,\beta}$.

3.1. Computation of the triple Bessel function integral $\mathcal{I}_{\alpha,\beta}$. To compute $\mathcal{I}_{\alpha,\beta}$ we use the formulas [16, 2.12.42 (11)–(13)] (see also [9, 6.578 (8)]), expressing it in terms of the associated Legendre functions (in the corresponding formulas [21, Chapter XIII, section 13·46(4), (5)] and [14, 10.22.72] there seems to be an error, an incorrect constant in the Q part). What we get splits naturally into the three cases below.

Case 1. $t < |x - z|$. Then $\mathcal{I}_{\alpha,\beta} \equiv 0$ (hence the whole kernel vanishes in this case).

Case 2. $|x - z| < t < x + z$. Then

$$\mathcal{I}_{\alpha,\beta} = \frac{1}{\sqrt{2\pi}} \frac{(xz)^{\alpha+\beta-1}}{t^{\alpha+\beta}} (\sin v)^{\alpha+\beta-1/2} \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(\cos v),$$

where $v \in (0, \pi)$ is such that

$$(3.2) \qquad \cos v = \frac{x^2 + z^2 - t^2}{2xz} \in (-1, 1),$$

and P is the Ferrers function of the first kind (the associated Legendre function of the first kind on the cut); cf. [14, Chapter 14].

Case 3. $t > x + z$. Then

$$\mathcal{I}_{\alpha,\beta} = \sqrt{\frac{2}{\pi^3}} \frac{(xz)^{\alpha+\beta-1}}{t^{\alpha+\beta}} (\sinh u)^{\alpha+\beta-1/2} \sin(\pi\beta) e^{\pi i(\alpha+\beta-1/2)} Q_{\alpha-1/2}^{1/2-\alpha-\beta}(\cosh u),$$

where $u > 0$ is such that

$$(3.3) \qquad \cosh u = \frac{t^2 - x^2 - z^2}{2xz} \in (1, \infty),$$

and Q is the associated Legendre function of the second kind; see [14, 14.3.7]. Since $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$ is not defined for $\beta = 1, 2, \ldots$, the formula above must be understood in a limiting sense. To overcome this inconvenience, instead of Q we rather use Olver's function (cf. [14, 14.3.10])

$$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = e^{\pi i(\alpha+\beta-1/2)} \frac{Q_{\alpha-1/2}^{1/2-\alpha-\beta}(y)}{\Gamma(1-\beta)},$$

which (unlike $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$) is, in our situation, always real-valued and defined for all α and β. This leads to

$$\mathcal{I}_{\alpha,\beta} = \mathcal{C}(\beta)\sqrt{\frac{2}{\pi^3}}\frac{(xz)^{\alpha+\beta-1}}{t^{\alpha+\beta}}(\sinh u)^{\alpha+\beta-1/2}\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(\cosh u),$$

with $\mathcal{C}(\beta) = \sin(\pi\beta)\Gamma(1-\beta)$ understood in the limiting sense when $\beta = 1,2,\ldots$. Observe that $\mathcal{C}(\beta) = 0$ if and only if $\beta = 0,-1,-2,\ldots$ and for such β this part of the kernel vanishes. Further, $\mathcal{C}(\beta) > 0$ when $\beta > 0$, and for $\beta < 0$ the sign of $\mathcal{C}(\beta)$ is $(-1)^{\lfloor\beta\rfloor}$ provided that $\beta \neq -1,-2,\ldots$. Observe that the constant in question can be written, with a limiting understanding, as $\mathcal{C}(\beta) = \pi/\Gamma(\beta)$, by Euler's reflection formula

$$(3.4) \qquad \sin(\pi y)\Gamma(1-y)\Gamma(y) = \pi.$$

Summing up Cases 1–3, one has
$$(3.5)$$

$$\mathcal{I}_{\alpha,\beta} = \frac{(xz)^{\alpha+\beta-1}}{\sqrt{2\pi}t^{\alpha+\beta}}\begin{cases} 0, & t < |x-z|, \\ (\sin v)^{\alpha+\beta-1/2}\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(\cos v), & |x-z| < t < x+z, \\ \frac{2}{\Gamma(\beta)}(\sinh u)^{\alpha+\beta-1/2}\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(\cosh u), & x+z < t, \end{cases}$$

where v and u are related to t, x, z by (3.2) and (3.3).

3.2. Explicit instances of Legendre functions $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ and $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$.

For some α and β, the functions $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ and $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ can be expressed in a more explicit way. We now derive some of these more elementary expressions. This is of importance for our further development, since we need to cover certain values of (α,β) for which asymptotics of $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ and $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ (see section 3.3 below) are in a sense singular and seem to be inaccessible in the standard literature on special functions. Independently, of course, it is of interest to know as explicit a form of the kernel as possible, at least for some α and β.

In what follows we always consider $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ on $(-1,1)$ and $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ on $(1,\infty)$. Also, we always assume $\alpha > -1$ and $\alpha + \beta > -1/2$, even though these assumptions can be weakened in some places below. We will use the formulas (cf. [14, 14.3.1, 14.3.19, 14.3.20])

$$(3.6)$$

$$\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \left(\frac{1+y}{1-y}\right)^{(1/2-\alpha-\beta)/2}\mathbf{F}\left(\alpha+\frac{1}{2},-\alpha+\frac{1}{2};\alpha+\beta+1/2;\frac{1-y}{2}\right),$$

$$(3.7)$$

$$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = 2^{\alpha-1/2}\Gamma\left(\alpha+\frac{1}{2}\right)\frac{(y+1)^{(1/2-\alpha-\beta)/2}}{(y-1)^{(\alpha-\beta+3/2)/2}}\mathbf{F}\left(\alpha+\frac{1}{2},1-\beta;2\alpha+1;\frac{2}{1-y}\right)$$

$$(3.8) \qquad = \frac{-\pi}{2\sin(\pi(\alpha+\beta-1/2))}$$

$$\times\left[\frac{1}{\Gamma(1-\beta)}\left(\frac{y-1}{y+1}\right)^{(\alpha+\beta-1/2)/2}\mathbf{F}\left(\frac{1}{2}-\alpha,\alpha+\frac{1}{2};\alpha+\beta+\frac{1}{2};\frac{1-y}{2}\right)\right.$$

$$\left.-\frac{1}{\Gamma(2\alpha+\beta)}\left(\frac{y+1}{y-1}\right)^{(\alpha+\beta-1/2)/2}\mathbf{F}\left(\frac{1}{2}-\alpha,\alpha+\frac{1}{2};\frac{3}{2}-\alpha-\beta;\frac{1-y}{2}\right)\right],$$

where $\mathbf{F}$ stands for Olver's hypergeometric function (see [14, sections 15.1, 15.2]),

$$\mathbf{F}(a, b; c; y) = \frac{1}{\Gamma(c)} \, {}_2F_1(a, b; c; y),$$

with ${}_2F_1$ being the Gauss hypergeometric function; note that these functions are symmetric in the first two parameters. The formula (3.7) has to be understood in a limiting sense for $\alpha = -1/2$. In (3.8) we assume that $\alpha + \beta - 1/2$ is not integer. Then, in cases when β is a positive integer or $2\alpha + \beta$ is a nonpositive integer, the formula has to be understood in a limiting sense. Furthermore, the following connection with the classical Jacobi polynomials, here denoted by $\mathbb{P}_m^{\gamma, \delta}$, will be used (cf. [14, 18.5.7], [11, p. 212]):

$$(3.9) \quad \mathbb{P}_m^{\gamma, \delta}(y) = \frac{\Gamma(m + \gamma + 1)}{m!} \mathbf{F}\left(-m, m + \gamma + \delta + 1; \gamma + 1; \frac{1 - y}{2}\right)$$

$$(3.10) \qquad = \frac{\Gamma(\gamma + \delta + 2m + 1)\Gamma(-2m - \gamma - \delta)}{m!\Gamma(\gamma + \delta + m + 1)} \left(\frac{y - 1}{2}\right)^m$$

$$\times \mathbf{F}\left(-m, -m - \gamma; -2m - \gamma - \delta; \frac{2}{1 - y}\right), \qquad m = 0, 1, 2, \ldots,$$

with suitable limiting understanding of the cases when singularities occur in (3.10). For the sake of clarity, we restrict our attention to $-1 < y < 1$ in (3.9) and to $y > 1$ in (3.10).

In the computations of $\mathsf{P}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ and $\mathsf{Q}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ below we distinguish three main cases.

Case 1. β is a nonpositive integer, say $\beta = -n$, $n = 0, 1, 2, \ldots$. Observe that in this situation always $\alpha > -1/2$. We focus on $\mathsf{P}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ only, since for the considered parameters the form of $\mathsf{Q}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ is irrelevant for the kernel (the part of the kernel expressed by $\mathsf{Q}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ vanishes for these parameters; see (3.5)).

Applying the linear transformation (cf. [14, 15.8.1])

$$\mathbf{F}(a, b; c; y) = (1 - y)^{c - a - b}\mathbf{F}(c - a, c - b; c; y)$$

to (3.6) and then using (3.9), we arrive at an expression for $\mathsf{P}_{\alpha-1/2}^{1/2 - \alpha - \beta}$ in terms of Jacobi (actually ultraspherical) polynomials,

$$(3.11) \quad \mathsf{P}_{\alpha-1/2}^{1/2 - \alpha - \beta}(y) = \frac{2^{n - \alpha + 1/2}n!}{\Gamma(\alpha + 1/2)}\left[(1 + y)(1 - y)\right]^{(\alpha - n - 1/2)/2}\mathbb{P}_n^{\alpha - n - 1/2, \alpha - n - 1/2}(y).$$

Note that in the special case of $n = 0 = \beta$ one has $\mathbb{P}_n^{\alpha - n - 1/2, \alpha - n - 1/2}(y) \equiv 1$.

Case 2. $2\alpha + \beta = 0$. Notice that in this case $-1 < \alpha < 1/2$. Using (3.6), (3.7), and the identity (cf. [14, 15.4.6]) $\mathbf{F}(a, b; b; y) = (1 - y)^{-a}/\Gamma(b)$, we find that

$$(3.12) \qquad \mathsf{P}_{\alpha-1/2}^{1/2 - \alpha - \beta}(y) = \frac{2^{\alpha + 1/2}}{\Gamma(1/2 - \alpha)}\left[(1 + y)(1 - y)\right]^{-(\alpha + 1/2)/2},$$

$$(3.13) \qquad \mathsf{Q}_{\alpha-1/2}^{1/2 - \alpha - \beta}(y) = \frac{\sqrt{\pi}}{2^{\alpha + 1/2}\Gamma(\alpha + 1)}\left[(y + 1)(y - 1)\right]^{-(\alpha + 1/2)/2}.$$

Here, in the computation related to $\mathsf{Q}_{\alpha-1/2}^{1/2 - \alpha - \beta}$, we used the duplication formula for the gamma function,

$$(3.14) \qquad \Gamma(y)\Gamma(y + 1/2) = \sqrt{\pi}2^{-2y + 1}\Gamma(2y).$$

Furthermore, in this computation we treated the value $\alpha = -1/2$ in a limiting sense.

Case 3. α is half of an odd integer, say $\alpha = n + 1/2$, $n = -1, 0, 1, 2, \ldots$. Note that in this situation $\beta > -n - 1$.

To begin, we first consider the subcase $n = -1$ separately. To this end, $\alpha = -1/2$ and $\beta > 0$. The formulas we then get are

$$(3.15) \qquad \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \frac{1}{\Gamma(\beta)} \left(\frac{1+y}{1-y} \right)^{(1-\beta)/2},$$

$$(3.16) \qquad \mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \frac{1}{2} \left[\left(\frac{y+1}{y-1} \right)^{(1-\beta)/2} + \left(\frac{y-1}{y+1} \right)^{(1-\beta)/2} \right].$$

To obtain (3.15), one observes that the first parameter of $\mathbf{F}$ in (3.6) vanishes and, consequently, this function is constant and equal to $1/\Gamma(\beta)$. As for (3.16), we use (3.7) and, with a limiting understanding, the duplication formula (3.14) with $y = \alpha + 1/2$ to see that

$$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \left(\frac{y+1}{y-1} \right)^{(1-\beta)/2} {}_2F_1 \left(\alpha + \frac{1}{2}, 1 - \beta; 2\alpha + 1; \frac{2}{1-y} \right) \bigg|_{\alpha=-1/2},$$

where the hypergeometric function is understood in a limiting sense. To find its explicit form we eliminate the singularity by means of the identity (cf. [14, 15.5.15])

$$2 \, {}_2F_1 \left(\alpha + \frac{1}{2}, 1 - \beta; 2\alpha + 1; \frac{2}{1-y} \right) = {}_2F_1 \left(\alpha + \frac{1}{2}, 1 - \beta; 2\alpha + 2; \frac{2}{1-y} \right)$$
$$+ {}_2F_1 \left(\alpha + \frac{3}{2}, 1 - \beta; 2\alpha + 2; \frac{2}{1-y} \right).$$

Letting $\alpha = -1/2$ on the right-hand side here and using the formula ${}_2F_1(a, b; a; y) = (1-y)^{-b}$ (essentially stated in Case 2 in terms of $\mathbf{F}$), we see that

$${}_2F_1 \left(\alpha + \frac{1}{2}, 1 - \beta; 2\alpha + 1; \frac{2}{1-y} \right) \bigg|_{\alpha=-1/2} = \frac{1}{2} \left[1 + \left(\frac{y+1}{y-1} \right)^{\beta-1} \right].$$

Now (3.16) follows.

From now to the end of Case 3, we assume $n \geq 0$. By (3.6) and (3.9) (recall that $\mathbf{F}$ is symmetric in the first two parameters) one easily gets

$$(3.17) \qquad \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \frac{n!}{\Gamma(2n+\beta+1)} \left(\frac{1+y}{1-y} \right)^{-(n+\beta)/2} \mathbb{P}_n^{n+\beta,-n-\beta}(y).$$

Finally, to compute $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ for $\alpha = n + 1/2$, we first assume that β is not integer and employ (3.8). Then, with the aid of (3.9),

$$(3.18)$$

$$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) = \frac{\pi n!}{2 \sin(\pi(n+\beta)) \Gamma(1-\beta) \Gamma(2n+\beta+1)}$$
$$\times \left[\left(\frac{y+1}{y-1} \right)^{(n+\beta)/2} \mathbb{P}_n^{-n-\beta,n+\beta}(y) - \left(\frac{y-1}{y+1} \right)^{(n+\beta)/2} \mathbb{P}_n^{n+\beta,-n-\beta}(y) \right].$$
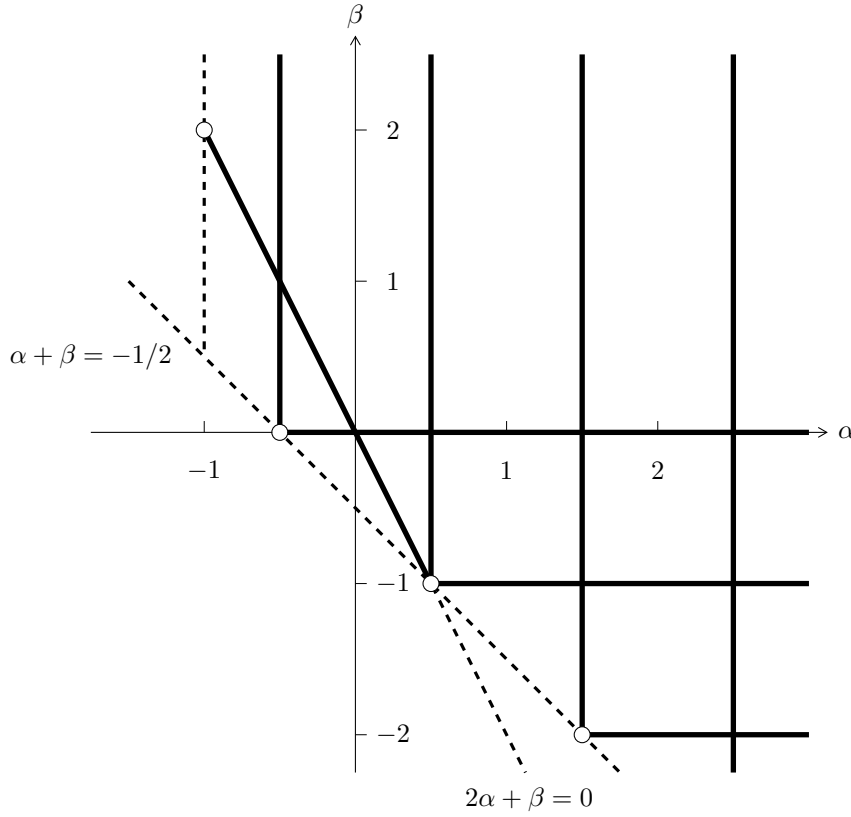
FIG. 1. *Cases when the kernel has explicit form.*

It remains to consider $\beta = m$, $m = 1, 2, \ldots$ (the case when β is a nonpositive integer is irrelevant for our purposes). In this situation, $\mathbf{Q}^{1/2-\alpha-\beta}_{\alpha-1/2}$ can be computed in fact for any α (recall that at the moment we are considering $\alpha \neq -1/2$). Indeed, using (3.7) and (3.10) we arrive at

$$(3.19) \qquad \mathbf{Q}^{1/2-\alpha-\beta}_{\alpha-1/2}(y) = \frac{\sqrt{\pi}(-1)^{m+1}(m-1)!\,\Gamma(2\alpha+1)}{2^{\alpha-m+3/2}\Gamma(\alpha+1)\Gamma(2\alpha+m)}$$
$$\times \left[(y+1)(y-1)\right]^{(1/2-\alpha-m)/2}\mathbb{P}^{1/2-\alpha-m,1/2-\alpha-m}_{m-1}(y).$$

To be precise, here we also used the duplication and reflection formulas (3.14) and (3.4). Note that for $\beta = 1$ and $\alpha = -1/2$ one has $\mathbf{Q}^{1/2-\alpha-\beta}_{\alpha-1/2}(y) \equiv 1$.

Remark 3.1. From the above considerations it follows that the kernel $K^{\alpha,\beta}_t(x,z)$ can be written explicitly, by means of elementary functions and possibly Jacobi polynomials, whenever $2\alpha + \beta = 0$ or $-\beta \in \mathbb{N}$ or $\alpha + 1/2 \in \mathbb{N}$; see Figure 1, where bold lines represent the cases when the kernel has the explicit form.

3.3. Asymptotics of $\mathbf{P}^{1/2-\alpha-\beta}_{\alpha-1/2}$ and $\mathbf{Q}^{1/2-\alpha-\beta}_{\alpha-1/2}$. To estimate the kernel our strategy will be to employ known asymptotics of the Legendre functions near singular points (-1^+ and 1^- in the case of $\mathsf{P}^{1/2-\alpha-\beta}_{\alpha-1/2}$, and 1^+ and ∞ in the case of $\mathbf{Q}^{1/2-\alpha-\beta}_{\alpha-1/2}$).
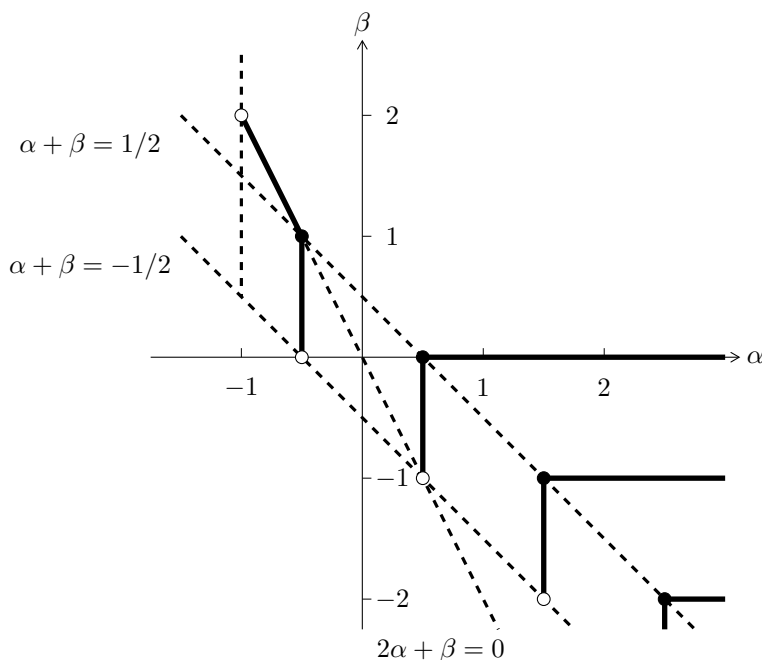
FIG. 2. *The exceptional set E^{P}.*

These asymptotics will be taken either from the literature (cf. [8, section 3.9.2], [11, Table 4.8.2], [14, section 14.8]) or from the explicit formulas derived in the previous section in case (α, β) belongs to one of the exceptional sets E^{P}, E^{Q} that will be defined in a moment. In the asymptotic expressions below we always write multiplicative constants, usually depending on α and β, when they may decide about signs.

Recall that we are considering $\alpha > -1$ and $\beta > -\alpha - 1/2$. The cases of singular points 1^- and ∞ are clear, and we have

$$(3.20) \qquad \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq (1-y)^{-(1/2-\alpha-\beta)/2}, \qquad y \to 1^-,$$

$$(3.21) \qquad \mathsf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq y^{-\alpha-1/2}, \qquad y \to \infty.$$

In order to treat the remaining two singular points, we define the exceptional sets

$$E^{\mathsf{P}} = \big\{(\alpha, \beta) : \alpha + \beta \geq 1/2 \ \text{and} \ [-\beta \in \mathbb{N} \ \text{or} \ 2\alpha + \beta = 0]\big\}$$
$$\cup \big\{(\alpha, \beta) : \alpha + \beta < 1/2 \ \text{and} \ \alpha + 1/2 \in \mathbb{N}\big\},$$
$$E^{\mathsf{Q}} = \big\{(\alpha, \beta) : \alpha + \beta \geq 1/2 \ \text{and} \ 2\alpha + \beta = 0\big\} \cup \big\{(\alpha, \beta) : \alpha + \beta < 1/2 \ \text{and} \ \beta = 1\big\}.$$

These sets are visualized in Figures 2 and 3 with bold lines and black dots.

At -1^+ it happens that

$$(3.22) \quad \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq \begin{cases} (1+y)^{(1/2-\alpha-\beta)/2} \frac{1}{\Gamma(2\alpha+\beta)\Gamma(\beta)}, & \alpha + \beta > 1/2, \\ -\log(1+y)\sin[\pi(1/2-\alpha)], & \alpha + \beta = 1/2, \\ (1+y)^{-(1/2-\alpha-\beta)/2}\sin[\pi(1/2-\alpha)], & \alpha + \beta < 1/2 \end{cases}$$

as $y \to -1^+$ provided that $(\alpha, \beta) \notin E^{\mathsf{P}}$. On the other hand, for $(\alpha, \beta) \in E^{\mathsf{P}}$ the asymptotic is different (or rather inverse; moreover, there are no logarithms when
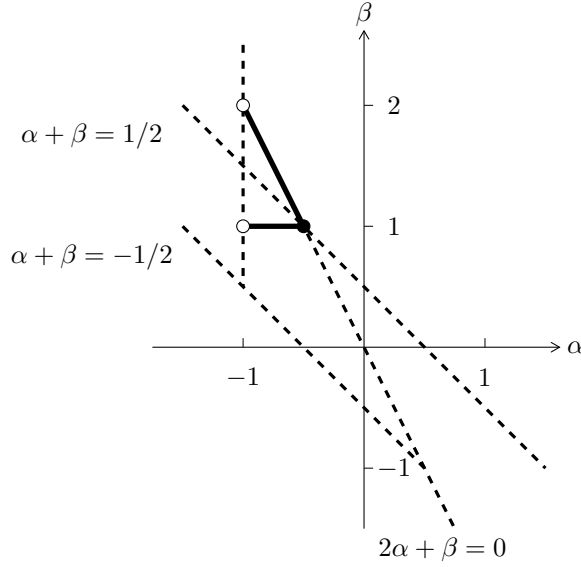
FIG. 3. *The exceptional set $E^{\mathbf{Q}}$.*

$\alpha + \beta = 1/2$). More precisely, for $(\alpha, \beta) \in E^{\mathbf{P}}$ we have

$$\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq \begin{cases} (1+y)^{(\alpha+\beta-1/2)/2} c_1(\alpha, \beta), & \alpha + \beta \geq 1/2, \\ (1+y)^{(1/2-\alpha-\beta)/2} c_2(\alpha, \beta), & \alpha + \beta < 1/2. \end{cases}$$

Here the constants c_1, c_2 are to indicate signs: $c_1 = 1$ if $2\alpha + \beta = 0$ and $c_1 = (-1)^\beta$ if $\beta = 0, -1, -2, \ldots$; $c_2 = 1$ if $\alpha = -1/2$ and $c_2 = (-1)^{\alpha-1/2}$ for $\alpha = 1/2, 3/2, 5/2, \ldots$.

Finally, as $y \to 1^+$ and $(\alpha, \beta) \notin E^{\mathbf{Q}}$, we have

(3.23) $$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq \begin{cases} (y-1)^{(1/2-\alpha-\beta)/2} \frac{1}{\Gamma(2\alpha+\beta)}, & \alpha + \beta > 1/2, \\ -\log(y-1) \frac{1}{\Gamma(\alpha+1/2)}, & \alpha + \beta = 1/2, \\ (y-1)^{(\alpha+\beta-1/2)/2} \frac{1}{\Gamma(1-\beta)}, & \alpha + \beta < 1/2, \end{cases}$$

whereas for $(\alpha, \beta) \in E^{\mathbf{Q}}$ (see also (3.19) with $m = 1 = \beta$) we have

$$\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \simeq \begin{cases} (y-1)^{(\alpha+\beta-1/2)/2}, & \alpha + \beta \geq 1/2, \\ (y-1)^{(1/2-\alpha-\beta)/2}, & \alpha + \beta < 1/2. \end{cases}$$

Neglecting signs, the asymptotics at -1^+ and 1^+ can be written in a more compact way, respectively,

$$\left| \mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \right| \simeq \begin{cases} (1+y)^{-|\alpha+\beta-1/2|/2}, & (\alpha, \beta) \notin E^{\mathbf{P}}, \ \alpha + \beta \neq 1/2, \\ -\log(1+y), & (\alpha, \beta) \notin E^{\mathbf{P}}, \ \alpha + \beta = 1/2, \\ (1+y)^{|\alpha+\beta-1/2|/2}, & (\alpha, \beta) \in E^{\mathbf{P}}, \end{cases}$$

and

$$\left| \mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}(y) \right| \simeq \begin{cases} (y-1)^{-|\alpha+\beta-1/2|/2}, & (\alpha, \beta) \notin E^{\mathbf{Q}}, \ \alpha + \beta \neq 1/2, \\ -\log(y-1), & (\alpha, \beta) \notin E^{\mathbf{Q}}, \ \alpha + \beta = 1/2, \\ (y-1)^{|\alpha+\beta-1/2|/2}, & (\alpha, \beta) \in E^{\mathbf{Q}}. \end{cases}$$
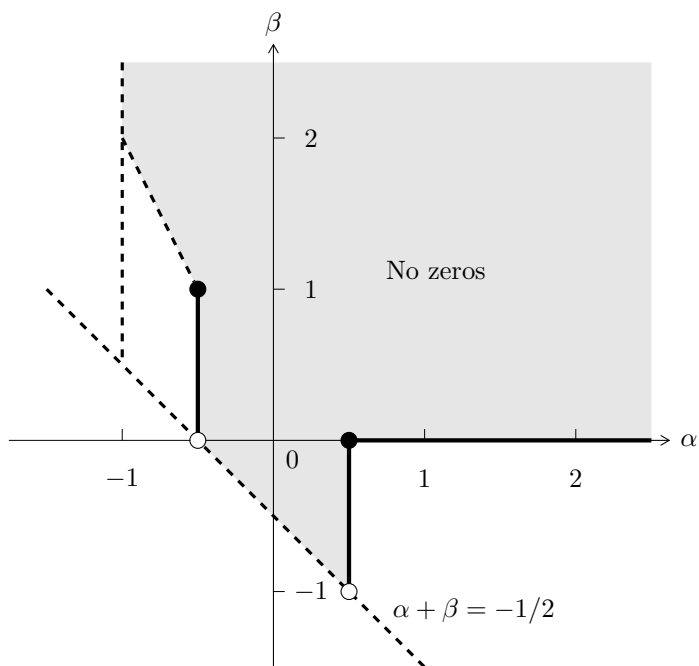
FIG. 4. *Zeros of* $P_{\alpha-1/2}^{1/2-\alpha-\beta}$ *in* $(-1,1)$.

3.4. Zeros of $P_{\alpha-1/2}^{1/2-\alpha-\beta}$ and $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$. We continue considering these functions on $(-1,1)$ and $(1,\infty)$, respectively. Since we are going to obtain possibly sharp estimates in terms of asymptotics of these functions, it is important to know whether, given α and β, they have zeros. If this is not the case, asymptotics invoked in section 3.3 (see (3.20) and (3.21)) imply that the functions are strictly positive.

We will prove the following.

PROPOSITION 3.2. *Let $\alpha > -1$ and $\alpha + \beta > -1/2$.*
(a) *The function $P_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(-1,1)$ if $\alpha \geq -1/2$ and $\beta \geq 0$, or $\alpha < -1/2$ and $\beta > -2\alpha$, or $\alpha \leq 1/2$ and $\beta < 0$. Otherwise, $P_{\alpha-1/2}^{1/2-\alpha-\beta}$ has at least one zero in $(-1,1)$.*
(b) *The function $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$ has exactly one zero if $\alpha < -1/2$ and $1 < \beta < -2\alpha$. Otherwise, $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(1,\infty)$.*

See Figures 4 and 5, where gray regions together with bold lines and black dots represent pairs (α,β) for which $P_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(-1,1)$ and $Q_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(1,\infty)$, respectively.

Proof of Proposition 3.2. We first prove (a). From the explicit formulas (3.11), (3.12), (3.15), and (3.17) we know, respectively, that there is no zero when $\beta = 0$ or $\beta = -2\alpha$ or $\alpha = -1/2$ or $[\alpha = 1/2$ and $\beta < 0]$. Further, by the first item (a) in [14, section 14.16(ii)], it follows that there is no zero if $\alpha, \beta > 0$ and $\alpha + \beta \geq 1/2$. Combining the same condition with [14, 14.9.5] we infer that no zeros may occur if $\alpha < 0$ and $\beta > -2\alpha$ and $\alpha + \beta \geq 1/2$. The lack of zeros in case $-1/2 < \alpha < 1/2$ and $\alpha + \beta < 1/2$ follows from item (c) in [14, section 14.16(ii)], combined with [14, 14.9.5]
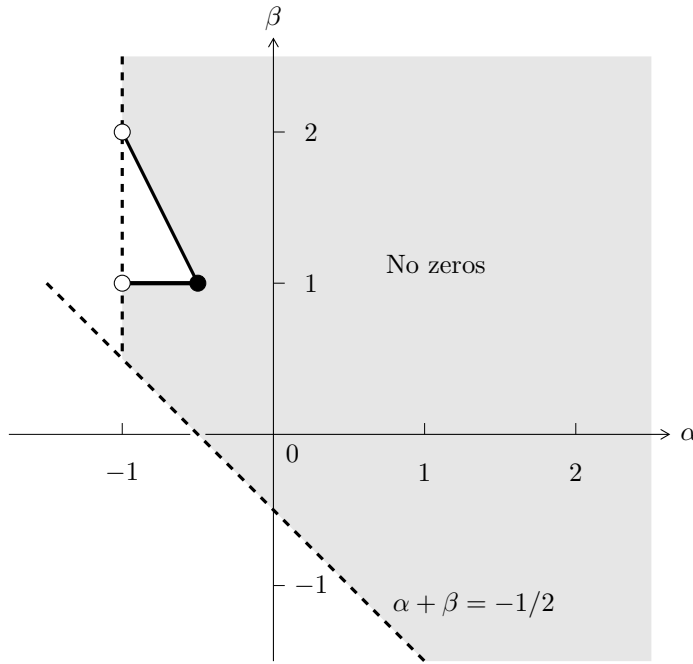
FIG. 5. *Zeros of* $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ *in* $(1, \infty)$.

when $\alpha < 0$ comes into play. Altogether, the above shows that there are no zeros for (α, β) indicated in item (a) of the proposition.

We now treat (α, β), for which there is at least one zero. The explicit formula (3.17) shows that this is the case when $\alpha = 1/2, 3/2, 5/2, \ldots$ and $\beta < 1/2 - \alpha$ (here we use the standard fact that Jacobi polynomials $\mathbb{P}_m^{\gamma,\delta}$ have m zeros in the interval $(-1, 1)$ whenever $\gamma, \delta > -1$). Further, according to the first item (a) in [14, section 14.16(ii)], there is at least one zero provided that $\alpha + \beta \geq 1/2$ and $\beta < 0$. The regions defined by the conditions $n + 1/2 < \alpha < n + 3/2$, $\alpha + \beta < 1/2$, $2\alpha + \beta > n + 1$, $n \in \mathbb{N}$, are covered by the first item (b) in [14, section 14.16(ii)], while the adjacent regions $n + 1/2 < \alpha < n + 3/2$, $2\alpha + \beta \leq n + 1$, $n \geq 1$, by the second item (a) in [14, section 14.16(ii)]. The remaining two regions defined by $\alpha < -1/2$, $\beta < -2\alpha$ and $1/2 < \alpha < 3/2$, $2\alpha + \beta \leq 1$, respectively, can be dealt with with the aid of the asymptotics from section 3.3. Indeed, taking into account (3.20), it is enough, for continuity reasons, to check that in the regions in question the asymptotic expressions at -1^+ are negative. But this immediately follows from (3.22).

Passing to (b), according to [14, section 14.16(iii)], $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(1, \infty)$ when $\alpha > -1/2$ and at most one zero when $\alpha < -1/2$. Moreover, by the explicit formulas (3.16), (3.13), and (3.19) we know, respectively, that no zeros occur when $\alpha = -1/2$, or $\alpha < -1/2$ and $\beta = -2\alpha$, or $\alpha < -1/2$ and $\beta = 1$. Further, the lack of zeros for $\alpha < -1/2$ and [$\beta < 1$ or $\beta > -2\alpha$] follows by applying Whipple's formula [14, 14.9.16] and then using the criterion from [14, section 14.16(iii)] for the associated Legendre function of the first kind. More precisely, this covers $\beta > -2\alpha$, and to treat $\beta < 1$ one combines the criterion just mentioned with the formula [14, 14.9.11]. Finally, there is one zero in the triangle $\alpha < -1/2$, $1 < \beta < -2\alpha$, in view of the asymptotic expressions (3.21) and (3.23). The conclusion follows. $\quad\square$

It is worth observing that the fact that $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ and $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ have no zeros in $(-1,1)$ and $(1,\infty)$, respectively, when $\alpha > -1/2$ and $\beta > 0$, is a straightforward consequence of (2.1) and (3.5).

3.5. Estimates of the kernel $K_t^{\alpha,\beta}(x,z)$. Using (3.1), (3.5), and the asymptotics from section 3.3, and taking into account continuity of the functions under consideration, we conclude the following estimates for the kernel.

Region $|x - z| < t < x + z$. If $(\alpha, \beta) \notin E^{\mathsf{P}}$, then

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{\beta-1}}{t^{2\alpha+2\beta}}\left[(\sin v)^2(1 - \cos v)\right]^{(\alpha+\beta-1/2)/2}$$
$$\times \begin{cases} (1 + \cos v)^{-(\alpha+\beta-1/2)/2}, & \alpha + \beta > 1/2, \\ 1 - \log(1 + \cos v), & \alpha + \beta = 1/2, \\ (1 + \cos v)^{(\alpha+\beta-1/2)/2}, & \alpha + \beta < 1/2, \end{cases}$$

whereas in case $(\alpha, \beta) \in E^{\mathsf{P}}$ one has a different and simpler (no logarithmic case) bound

$$|K_t^{\alpha,\beta}(x,z)|$$
$$\lesssim \frac{(xz)^{\beta-1}}{t^{2\alpha+2\beta}}\left[(\sin v)^2(1 - \cos v)\right]^{(\alpha+\beta-1/2)/2} \begin{cases} (1 + \cos v)^{(\alpha+\beta-1/2)/2}, & \alpha + \beta \geq 1/2, \\ (1 + \cos v)^{-(\alpha+\beta-1/2)/2}, & \alpha + \beta < 1/2. \end{cases}$$

Here v is determined by (3.2). For those (α, β) for which $\mathsf{P}_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(-1,1)$ (see Proposition 3.2), the estimates are sharp, one can replace $\lesssim$ by $\simeq$, and, moreover, one can suppress the absolute value of the kernel. Otherwise, the estimates are sharp provided that $\cos v$ is in a (sufficiently small) neighborhood of -1 or 1.

Region $x + z < t$. If $(\alpha, \beta) \notin E^{\mathbf{Q}}$ and in addition $-\beta \notin \mathbb{N}$, then

$$|(-1)^{\lfloor \beta \wedge 0 \rfloor} K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{\beta-1}}{t^{2\alpha+2\beta}}(\sinh u)^{\alpha+\beta-1/2}(\cosh u + 1)^{-\alpha-1/2}$$
$$\times \begin{cases} \left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{-(\alpha+\beta-1/2)/2}, & \alpha + \beta > 1/2, \\ 1 - \log\left(\frac{\cosh u - 1}{\cosh u + 1}\right), & \alpha + \beta = 1/2, \\ \left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{(\alpha+\beta-1/2)/2}, & \alpha + \beta < 1/2, \end{cases}$$

and in case $(\alpha, \beta) \in E^{\mathbf{Q}}$ we have

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{\beta-1}}{t^{2\alpha+2\beta}}(\sinh u)^{\alpha+\beta-1/2}(\cosh u + 1)^{-\alpha-1/2}$$
$$\times \begin{cases} \left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{(\alpha+\beta-1/2)/2}, & \alpha + \beta \geq 1/2, \\ \left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{-(\alpha+\beta-1/2)/2}, & \alpha + \beta < 1/2. \end{cases}$$

Here u is determined by (3.3). For those (α, β) for which $\mathbf{Q}_{\alpha-1/2}^{1/2-\alpha-\beta}$ has no zeros in $(1,\infty)$ (see Proposition 3.2), the estimates are sharp, one can replace $\lesssim$ by $\simeq$, and, moreover, one can suppress the absolute values. Otherwise, the estimates are sharp provided that $\cosh v$ is sufficiently large or sufficiently close to 1.

From the above bounds we can readily get estimates of $K_t^{\alpha,\beta}(x,z)$ in terms of t, x, z. To do that, we use the following identities that hold when $|x - z| < t$:

$$1 + \cos v = \cosh u - 1 = \frac{1}{2}\frac{|(x+z)^2 - t^2|}{xz},$$

$$1 - \cos v = \cosh u + 1 = \frac{1}{2}\frac{t^2 - (x-z)^2}{xz},$$

$$\sin^2 v = \sinh^2 u = \frac{1}{4}\frac{|(x+z)^2 - t^2|}{xz}\frac{t^2 - (x-z)^2}{xz}.$$

As the outcome, also taking into account (2.3), we obtain the main result concerning pointwise estimates of $K_t^{\alpha,\beta}(x,z)$.

THEOREM 3.3. *Assume that* $\alpha > -1$ *and* $\alpha + \beta > -1/2$. *Let* $t, x, z > 0$.
(1) *The kernel* $K_t^{\alpha,\beta}(x,z)$ *vanishes when* $t < |x - z|$.
(2) *The following estimates hold uniformly in* $|x - z| < t < x + z$:
  (2a) *If* $-\beta \in \mathbb{N}$ *or* $2\alpha + \beta = 0$, *then*

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{-\alpha-1/2}}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\alpha+\beta-1/2}\left(\frac{(x+z)^2 - t^2}{xz}\right)^{\alpha+\beta-\frac{1}{2}}.$$

  (2b) *If* $\alpha + 1/2 \in \mathbb{N}$, *then*

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{-\alpha-1/2}}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\alpha+\beta-1/2}.$$

  (2c) *For all* (α,β) *not covered by items* (2a) *and* (2b),

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{(xz)^{-\alpha-1/2}}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\alpha+\beta-1/2}$$

$$\times \begin{cases} \left(\frac{(x+z)^2 - t^2}{xz}\right)^{\alpha+\beta-1/2}, & \alpha + \beta < 1/2, \\ 1 + \log\left(\frac{4xz}{(x+z)^2 - t^2}\right), & \alpha + \beta = 1/2, \\ 1, & \alpha + \beta > 1/2. \end{cases}$$

  The absolute values in (2a)–(2c) *can be suppressed and* $\lesssim$ *can be replaced by* $\simeq$ *if and only if* (α,β) *satisfy neither* $\alpha > 1/2$ *and* $\beta < 0$ *nor* $\alpha < -1/2$ *and* $\beta < -2\alpha$.
(3) *The following bounds are uniform in* $t > x + z$:
  (3a) *If* $-\beta \in \mathbb{N}$, *then this part of the kernel vanishes.*
  (3b) *If* $2\alpha + \beta = 0$ *and* $\beta \neq 0$, *then*

$$K_t^{\alpha,\beta}(x,z) \simeq \frac{1}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\beta-1}\left(\frac{t^2 - (x+z)^2}{t^2 - (x-z)^2}\right)^{\alpha+\beta-1/2}.$$

  (3c) *If* $\beta = 1$, *then*

$$K_t^{\alpha,\beta}(x,z) \simeq \frac{1}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\beta-1}.$$

  (3d) *For all* (α,β) *not covered by items* (3a)–(3c),

$$|K_t^{\alpha,\beta}(x,z)| \lesssim \frac{1}{t^{2\alpha+2\beta}}\left[t^2 - (x-z)^2\right]^{\beta-1}$$

$$\times \begin{cases} \left(\frac{t^2-(x+z)^2}{t^2-(x-z)^2}\right)^{\alpha+\beta-1/2}, & \alpha+\beta < 1/2, \\ 1 + \log\left(\frac{t^2-(x-z)^2}{t^2-(x+z)^2}\right), & \alpha+\beta = 1/2, \\ 1, & \alpha+\beta > 1/2. \end{cases}$$

The relation $\lesssim$ in (3d) can be replaced by $\simeq$ if and only if (α,β) does not satisfy $1 < \beta < -2\alpha$. If this is the case, then also the absolute value can be suppressed provided that the kernel is multiplied by $(-1)^{\lfloor \beta \wedge 0 \rfloor}$.

Note that some estimates in Theorem 3.3 can be written in a simpler way by plugging in specific values of the parameters; nevertheless, we keep the general formulas for the sake of better comparison between the cases. Further, in all the cases the estimates are sharp ($\lesssim$ can be replaced by $\simeq$) provided that (t,x,z) are restricted to certain regions. More precisely, this happens when t is sufficiently close to $|x-z|$ or $x+z$ or ∞, that is,

$$\operatorname{dist}(t,|x-z|) < \varepsilon\sqrt{xz} \quad \text{or} \quad \operatorname{dist}(t,x+z) < \varepsilon\sqrt{xz} \quad \text{or} \quad t > \varepsilon^{-1}\sqrt{xz}$$

with $\varepsilon > 0$ small enough.

4. L^2-boundedness of the integral operator $M_t^{\alpha,\beta}$. The estimates of Theorem 3.3 allow us to verify directly the L^2-boundedness of the integral operator $M_t^{\alpha,\beta}$.

PROPOSITION 4.1. *Let $\alpha > -1$ and $\alpha + \beta > -1/2$. Then, for each $t > 0$, $L^2(d\mu_\alpha) \subset \operatorname{Dom} M_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$ is bounded on $L^2(d\mu_\alpha)$.*

This together with Proposition 1.1 implies the following.

COROLLARY 4.2. *Let $\alpha > -1$ and $\alpha + \beta > -1/2$. Then the operators $\mathcal{M}_t^{\alpha,\beta}$ and $M_t^{\alpha,\beta}$ coincide on $L^2(d\mu_\alpha)$.*

Proof of Proposition 4.1. In view of the scaling property of the kernel (1.2), we may assume $t = 1$. Then the estimates of Theorem 3.3 imply the following bound uniform in x and z:

$$|K_1^{\alpha,\beta}(x,z)| \lesssim \begin{cases} 0, & 1 < |x-z|, \\ (xz)^{-\alpha-1/2}[1-(x-z)^2]^\gamma\left(\frac{(x+z)^2-1}{xz}\right)^\gamma, & |x-z| < 1 < x+z, \\ [1-(x-z)^2]^{-\alpha-1/2}[1-(x+z)^2]^\gamma, & x+z < 1, \end{cases}$$

where $\gamma \in (-1,0)$ is a constant depending on α and β; actually, taking $\gamma = (\alpha + \beta - 1/2) \wedge (-\varepsilon)$ with a small $\varepsilon > 0$ will suffice (ε to take care of the logarithms; otherwise, $\varepsilon = 0$ would be enough). The right-hand side above can be simplified by taking into account the constraints on x and z and the relations $4xz = (x+z)^2 - (x-z)^2 \simeq (x+z)(x+z-|x-z|)$. We get

$$|K_1^{\alpha,\beta}(x,z)| \lesssim \begin{cases} 0, & |x-z| > 1, \\ (xz)^{-\alpha-1/2}\left(\frac{(1-|x-z|)(x+z-1)}{x+z-|x-z|}\right)^\gamma, & |x-z| < 1, \ x+z > 1, \\ (1-|x-z|)^{-\alpha-1/2}[1-(x+z)]^\gamma, & x+z < 1. \end{cases}$$

We will consider here separately the two integral operators defined by the expressions on the right-hand side. It is enough to verify that each of them is bounded on $L^2(d\mu_\alpha)$.

Let

$$L(x,z) = \chi_{\{|x-z|<1,\ x+z>1\}}(xz)^{-\alpha-1/2}\left(\frac{(1-|x-z|)(x+z-1)}{x+z-|x-z|}\right)^\gamma,$$

$$M(x, z) = \chi_{\{x+z<1\}}(1 - |x - z|)^{-\alpha - 1/2}[1 - (x + z)]^{\gamma},$$

and denote the corresponding integral operators (integration with respect to $d\mu_\alpha$) by $\mathbf{L}$ and $\mathbf{M}$, respectively. Our strategy to show $L^2(d\mu_\alpha)$-boundedness of $\mathbf{L}$ and $\mathbf{M}$ is mainly based on the Schur test, applied to integral operators $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{M}}$ defined by the kernels $(xz)^{\alpha+1/2}L(x, z)$ and $(xz)^{\alpha+1/2}M(x, z)$, respectively, integration being with respect to Lebesgue measure dx in $(0, \infty)$. Observe that $\mathbf{L}$ is bounded in $L^2(d\mu_\alpha)$ if and only if $\widetilde{\mathbf{L}}$ is bounded in $L^2(dx)$, and similarly for $\mathbf{M}$ and $\widetilde{\mathbf{M}}$. Recall that, taking into account the positivity and symmetry of our kernels, the Schur test says that the bound

$$\int_0^\infty (xz)^{\alpha+1/2} L(x, z)\, dz \lesssim 1, \qquad x > 0,$$

implies $L^p(dx)$-boundedness, $1 \le p \le \infty$, of $\widetilde{\mathbf{L}}$; an analogous implication holds for $\widetilde{\mathbf{M}}$.

To proceed, we first focus on $\mathbf{L}$, or rather $\widetilde{\mathbf{L}}$. We have (recall that $-1 < \gamma < 0$)

$$(xz)^{\alpha+1/2}L(x, z) = \chi_{\{|x-z|<1,\, x+z>1\}} \left(\frac{(1 - |x - z|)(x + z - 1)}{(1 - |x - z|) + (x + z - 1)} \right)^{\gamma}$$

$$\simeq \chi_{\{|x-z|<1,\, x+z>1\}} \left[(1 - |x - z|)^{\gamma} + (x + z - 1)^{\gamma} \right]$$

$$\lesssim \chi_{\{|x-z|<1\}}(1 - |x - z|)^{\gamma} + \chi_{\{|x-z|<1,\, x+z>1\}}(x + z - 1)^{\gamma}.$$

In the last sum the first term is given by an integrable convolution kernel, so it defines an operator bounded on all $L^p(dx)$, $1 \le p \le \infty$ (clearly, the Schur test applies as well with the same conclusion). To deal with the second term, we write

$$\int_0^\infty \chi_{\{|x-z|<1,\, x+z>1\}}(x + z - 1)^{\gamma}\, dz = \int_{|x-1|}^{x+1} (x + z - 1)^{\gamma}\, dz$$

$$\simeq x^{\gamma+1} - [(x - 1) \vee 0]^{\gamma+1} \lesssim 1, \qquad x > 0,$$

and invoke the Schur test. The $L^p(dx)$-boundedness of $\widetilde{\mathbf{L}}$ follows, and this implies the $L^2(d\mu_\alpha)$-boundedness of $\mathbf{L}$.

Next, we analyze $\mathbf{M}$. Here the Schur test leads to the conclusion when applied to $\widetilde{\mathbf{M}}$, but with $-1 < \alpha < -1/2$ excluded. Therefore, to cover all $\alpha > -1$, we argue in a more subtle way. In the first step we will show that the integral operator

$$\mathbf{M}_1 f(x) = \int_0^\infty \chi_{\{|x-z|\le 1/2\}} M(x, z) f(z)\, d\mu_\alpha(z)$$

is bounded on $L^2(d\mu_\alpha)$. Indeed, this follows by using the Schur test, since

$$\int_0^\infty \chi_{\{|x-z|\le 1/2\}} M(x, z)\, d\mu_\alpha(z) \simeq \int_0^\infty \chi_{\{|x-z|\le 1/2,\, x+z<1\}}(1 - x - z)^{\gamma} z^{2\alpha+1}\, dz$$

$$\le \chi_{\{x<3/4\}}(1 - x)^{\gamma+2\alpha+2} \int_0^1 (1 - s)^{\gamma} s^{2\alpha+1}\, ds$$

$$\lesssim 1, \qquad x > 0.$$

It remains to verify that

$$\mathbf{M}_2 f(x) = \int_0^\infty \chi_{\{x>z\}} \chi_{\{|x-z|>1/2\}} M(x, z) f(z)\, d\mu_\alpha(z)$$

is bounded in $L^2(d\mu_\alpha)$, because then automatically the same is true for its dual $\mathbf{M}_2^*$ and $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_2^*$.

Observe that the kernel of $\mathbf{M}_2$ can be estimated as

$$\chi_{\{x>z\}}\chi_{\{|x-z|>1/2\}}M(x,z) \lesssim \chi_{\{x>z,\,1/2<x<1,\,z<1-x\}}(1-x)^{-\alpha-1/2}(1-x-z)^\gamma,$$

so it is enough to check the bound

$$\left\|\chi_{\{1/2<x<1\}}(1-x)^{-\alpha-1/2}\int_0^\infty \chi_{\{z<1-x\}}(1-x-z)^\gamma f(z)z^{2\alpha+1}\,dz\right\|_{L^2(\mathbb{R}_+,dx)}$$
$$\lesssim \|x^{\alpha+1/2}f(x)\|_{L^2(\mathbb{R}_+,dx)}, \qquad f \in L^2(d\mu_\alpha).$$

By changing the variable $1 - x = y$ and letting $F(z) = z^{\alpha+1/2}f(z)$, we see that this task will be done once we justify that

$$\left\|\chi_{\{y<1\}}\int_0^y \left(\frac{z}{y}\right)^{\alpha+1/2}(y-z)^\gamma F(z)\,dz\right\|_{L^2(\mathbb{R}_+,dy)} \lesssim \|F\|_{L^2(\mathbb{R}_+,dx)}, \qquad F \in L^2(\mathbb{R}_+,dx).$$

Let $\mathcal{I}$ denote the left-hand side in the above estimate. Changing the variable $z = yr$ and then using Minkowski's integral inequality, we get

$$\mathcal{I} = \left\|\chi_{\{y<1\}}y^{\gamma+1}\int_0^1 r^{\alpha+1/2}(1-r)^\gamma F(yr)\,dr\right\|_{L^2(\mathbb{R}_+,dy)}$$
$$\leq \int_0^1 r^{\alpha+1/2}(1-r)^\gamma\left\|\chi_{\{y<1\}}y^{\gamma+1}F(yr)\right\|_{L^2(\mathbb{R}_+,dy)}\,dr.$$

To estimate the norm expression under the last integral, we change back the variable $yr = z$ and obtain

$$\left\|\chi_{\{y<1\}}y^{\gamma+1}F(yr)\right\|_{L^2(\mathbb{R}_+,dy)}$$
$$= \frac{1}{\sqrt{r}}\left(\int_0^r \left(\frac{z}{r}\right)^{2(\gamma+1)}|F(z)|^2\,dz\right)^{1/2} \leq \frac{1}{\sqrt{r}}\|F\|_{L^2(\mathbb{R}_+,dz)},$$

since $2(\gamma+1) > 0$. Consequently,

$$\mathcal{I} \leq \|F\|_{L^2(\mathbb{R}_+,dz)}\int_0^1 r^\alpha(1-r)^\gamma\,dr \lesssim \|F\|_{L^2(\mathbb{R}_+,dz)}.$$

This finishes proving $L^2(d\mu_\alpha)$-boundedness of $\mathbf{M}_2$ and thus also of $\mathbf{M}$.

The proof of Proposition 4.1 is now complete. $\qquad\Box$

5. Time variable norm estimates of the kernel $K_t^{\alpha,\beta}(x,z)$. In this section we find possibly sharp estimates (which in fact are sharp in some cases and presumably sharp in all cases) of the norm of $K_t^{\alpha,\beta}(x,z)$ in $L^r(t^\rho dt)$. We consider here all power weights t^ρ, $\rho \in \mathbb{R}$, and $1 \leq r < \infty$. The case $r = \infty$ is not treated. It corresponds to the maximal operator associated with radial spherical means, whose analysis requires more subtle methods than those we apply for $r < \infty$; in particular, evaluating first the supremum in t of the kernel does not lead to satisfactory results.

THEOREM 5.1. *Assume that $\alpha > -1$ and $\alpha + \beta > -1/2$. Let $1 \leq r < \infty$, and assume further that*

$$(5.1) \qquad \alpha + \beta > \frac{1}{2} - \frac{1}{r} \qquad and \qquad \left[\frac{\rho + 1}{r} < 2\alpha + 2 \ \ if \ -\beta \notin \mathbb{N} \right].$$

Then, uniformly in $x, z > 0$,

$$\left\| K_t^{\alpha,\beta}(x,z) \right\|_{L^r(t^\rho dt)} \lesssim (x+z)^{-2\alpha-1} \begin{cases} |x-z|^{(\rho+1)/r-1}, & \frac{\rho+1}{r} < 1 \\ 1 + \log^{1/r}\left(\frac{x+z}{|x-z|} \right), & \frac{\rho+1}{r} = 1 \\ (x+z)^{(\rho+1)/r-1}, & \frac{\rho+1}{r} > 1 \end{cases}$$

$$\times \begin{cases} 1, & \beta + \frac{1}{r} > 1 \\ 1 + \chi_{\{\beta \neq 0\}} \log^{1/r}\left(\frac{x}{z} \vee \frac{z}{x} \right), & \beta + \frac{1}{r} = 1 \\ \left(\frac{x}{z} \wedge \frac{z}{x} \right)^{\beta+1/r-1}, & \beta + \frac{1}{r} < 1 \end{cases}.$$

Moreover, the relation $\lesssim$ can be replaced by $\simeq$ in the above estimate if any of the following statements is true:

(a) *α and β satisfy neither $[\alpha < -1/2$ and $\beta < -2\alpha]$ nor $[\alpha > 1/2$ and $\beta < 0]$;*
(b) *α and β do not satisfy $[1 < \beta < -2\alpha$ or $-\beta \in \mathbb{N}]$, and x, z stay noncomparable;*
(c) *α and β do not satisfy $[1 < \beta < -2\alpha$ or $-\beta \in \mathbb{N}]$ and $(\rho+1)/r > 1$.*

Furthermore, if either of the conditions in (5.1) is not satisfied, then the $L^r(t^\rho dt)$ norm of the kernel is infinite.

Observe that there are some α, β for which the unweighted norm ($\rho = 0$) is infinite. This motivates the introduction of power weights.

In the proof of Theorem 5.1 we will use repeatedly the lemma below.

LEMMA 5.2. *Let $\gamma > -1$, $\delta \in \mathbb{R}$, and $0 < C < 1$ be fixed.*
(a) *The following relation holds uniformly in $0 < A \leq C$:*

$$\int_0^A w^\gamma (1-w)^\delta \, dw \simeq A^{\gamma+1}.$$

(b) *The following bounds hold uniformly in $C \leq A < 1$:*

$$\int_0^A w^\gamma (1-w)^\delta \, dw \simeq \begin{cases} 1, & \delta > -1, \\ \log \frac{1}{1-A}, & \delta = -1, \\ (1-A)^{\delta+1}, & \delta < -1. \end{cases}$$

For $\gamma \leq -1$, the integral diverges to infinity for any $0 < A < 1$.

Proof. This is a simple exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Theorem 5.1. We will integrate against $t^\rho dt$ the right-hand sides of the bounds in Theorem 3.3 raised to power r. We split this integration with respect to the four regions

$$|x - z| < t < \sqrt{x^2 + z^2} < t < x + z < t < \sqrt{2}(x+z) < t$$

and denote the resulting (nonnegative) integrals by $I_1, I_2, I_3,$ and I_4, respectively. These integrals, in general, will be analyzed separately. Then the resulting estimates will be merged via considering comparable and noncomparable values of x and z.

We proceed by considering the most involved situation when items (2c) and (3d) from Theorem 3.3 are combined. It is convenient to distinguish three main cases emerging naturally from the estimates in Theorem 3.3(2c) and (3d).

Case 1. $\alpha + \beta > 1/2$. We have

$$I_1 = (xz)^{-(\alpha+1/2)r} \int_{|x-z|}^{\sqrt{x^2+z^2}} \left[t^2 - (x-z)^2\right]^{(\alpha+\beta-1/2)r} t^{\rho-2(\alpha+\beta)r} \, dt$$

$$(5.2) \qquad = \frac{1}{2}(xz)^{-(\alpha+1/2)r}|x-z|^{\rho+1-r} \int_0^{\frac{2xz}{x^2+z^2}} w^{(\alpha+\beta-1/2)r}(1-w)^{(r-\rho-3)/2} \, dw,$$

where the second identity follows by the change of variables $t^2 = (x-z)^2/(1-w)$. Next, observing that on the interval of integration in I_2 one has $t \simeq x + z$ and $t^2 - (x-z)^2 \simeq xz$, and the length of that interval is $x + z - \sqrt{x^2 + z^2} \simeq xz/(x+z)$, we immediately get

$$I_2 \simeq (x+z)^{-2(\alpha+\beta)r+\rho-1}(xz)^{(\beta-1)r+1}.$$

Finally, changing the variable $t^2 = (x-z)^2/w$ we arrive at

$$I_3 + I_4 = \int_{x+z}^{\infty} \left[t^2 - (x-z)^2\right]^{(\beta-1)r} t^{\rho-2(\alpha+\beta)r} \, dt$$

$$(5.3) \qquad = \frac{1}{2}|x-z|^{-2(\alpha+1)r+\rho+1} \int_0^{\left(\frac{|x-z|}{x+z}\right)^2} w^{(\alpha+1)r+(1-\rho)/2-2}(1-w)^{(\beta-1)r} \, dw.$$

Consider now comparable x and z. In this situation, $xz \simeq (x+z)^2$. Further, since the upper limit of integration in (5.2) is separated from 0, by Lemma 5.2(b) we conclude that

$$(5.4) \qquad I_1 \simeq (x+z)^{-(2\alpha+1)r}|x-z|^{\rho+1-r} \begin{cases} 1, & r > \rho+1, \\ 1 + \log\frac{x+z}{|x-z|}, & r = \rho+1, \\ \left(\frac{x+z}{|x-z|}\right)^{\rho+1-r}, & r < \rho+1. \end{cases}$$

Since now

$$I_2 \simeq (x+z)^{-(2\alpha+1)r+\rho+1-r},$$

it is straightforward to see that I_2 is controlled by I_1. As for $I_3 + I_4$, we observe that the upper limit of integration in (5.3) is separated from 1 and apply Lemma 5.2(a) to get

$$I_3 + I_4 \simeq (x+z)^{-(2\alpha+1)r+\rho+1-r}.$$

Here the right-hand side is the same as in the case of I_2. Summing up, I_1 dominates $I_2 + I_3 + I_4$, and the desired bound for $x \simeq z$ follows.

Now let x and z be noncomparable. For symmetry reasons, we may assume that $x \gg z$. Then $x + z \simeq |x - z| \simeq x$. Taking into account that the upper limit of integration in (5.2) is separated from 1 and applying Lemma 5.2(a), we obtain

$$(5.5) \qquad I_1 \simeq x^{-(2\alpha+1)r} x^{\rho+1-r} \left(\frac{z}{x}\right)^{(\beta-1)r+1}.$$

Moreover, it is straightforward to see that I_2 is comparable to the right-hand side here. Passing to $I_3 + I_4$, we note that the upper limit of integration in (5.3) is now

separated from 0, so Lemma 5.2(b) can be applied. This leads to

$$I_3 + I_4 \simeq x^{-(2\alpha+1)r} x^{\rho+1-r} \begin{cases} 1, & \frac{1}{r} > 1 - \beta, \\ 1 + \log \frac{x}{z}, & \frac{1}{r} = 1 - \beta, \\ \left(\frac{z}{x}\right)^{(\beta-1)r+1}, & \frac{1}{r} < 1 - \beta. \end{cases}$$

Since, as easily verified, $I_3 + I_4$ controls $I_1 + I_2$, we get the bound asserted in the theorem for $x \gg z$. The conclusion follows.

Case 2. $\alpha + \beta < 1/2$. Since on the interval of integration in I_1 we have $(x+z)^2 - t^2 \simeq xz$, we get

$$I_1 \simeq (xz)^{-(\alpha+1/2)r} \int_{|x-z|}^{\sqrt{x^2+z^2}} \left[t^2 - (x-z)^2\right]^{(\alpha+\beta-1/2)r} t^{\rho-2(\alpha+\beta)r} \, dt.$$

The right-hand side here corresponds to I_1 from Case 1, so all the bounds for the present I_1 will be as in Case 1. Considering I_2, we observe that on the interval of integration $t^2 - (x-z)^2 \simeq xz$ and $t \simeq x + z$. Consequently,

$$I_2 \simeq (xz)^{-(\alpha+1/2)r} (x+z)^{\rho-2(\alpha+\beta)r-1} \int_{\sqrt{x^2+z^2}}^{x+z} \left[(x+z)^2 - t^2\right]^{(\alpha+\beta-1/2)r} t \, dt$$

$$(5.6) \qquad = \frac{1}{2} (xz)^{-(\alpha+1/2)r} (x+z)^{\rho+1-r} \int_0^{\frac{2xz}{(x+z)^2}} w^{(\alpha+\beta-1/2)r} \, dw,$$

where the last identity follows by the change of variables $t^2 = (x+z)^2(1-w)$. For I_3, we notice that on the interval of integration $t \simeq x + z$, and we write

$$I_3 \simeq (x+z)^{-2(\alpha+\beta)r+\rho-1}$$

$$\times \int_{x+z}^{\sqrt{2}(x+z)} \left[t^2 - (x-z)^2\right]^{-(\alpha+1/2)r} \left[t^2 - (x+z)^2\right]^{(\alpha+\beta-1/2)r} t \, dt$$

$$(5.7) \qquad \simeq (x+z)^{-(2\alpha+1)r+\rho+1-r} \left(\frac{xz}{(x+z)^2}\right)^{(\beta-1)r+1}$$

$$\times \int_0^{\frac{(x+z)^2}{(x+z)^2+4xz}} w^{(\alpha+\beta-1/2)r} (1-w)^{(1-\beta)r-2} \, dw,$$

where the last relation is obtained by changing the variable

$$(5.8) \qquad t^2 = (x+z)^2 + 4xz \frac{w}{1-w}.$$

Finally, on the interval of integration of I_4 one has $t^2 - (x-z)^2 \simeq t^2 - (x+z)^2 \simeq t^2$ and therefore

$$I_4 \simeq \int_{\sqrt{2}(x+z)}^\infty t^{-(2\alpha+1)r+\rho-r} \, dt \simeq (x+z)^{-(2\alpha+1)r+\rho+1-r}.$$

Now let x and z be comparable. Then I_1 satisfies (5.4). Further, the upper limit of integration in (5.6) is separated from 0, and hence that integral has the size of a (positive and finite) constant. Therefore, using also $xz \simeq (x+z)^2$,

$$I_2 \simeq (x+z)^{-(2\alpha+1)r+\rho+1-r},$$

which, as in Case 1, is controlled by I_1. Clearly, the latter is also true for I_4. As for I_3, the upper limit of integration is separated both from 0 and 1, so we get the same behavior as for I_2 and I_4. Altogether, this gives the relevant bound for $x \simeq z$.

When x and z are noncomparable, say $x \gg z$, we argue in a similar way. I_1 satisfies (5.5). To estimate I_2 we just integrate in (5.6), getting the same behavior as for I_1, so

$$I_1 + I_2 \simeq x^{-(2\alpha+1)r} x^{\rho+1-r} \left(\frac{z}{x}\right)^{(\beta-1)r+1}.$$

The upper limit of integration in (8) is separated from 0, so applying Lemma 5.2(b) we arrive at

$$I_3 \simeq x^{-(2\alpha+1)r} x^{\rho+1-r} \begin{cases} 1, & \frac{1}{r} > 1 - \beta, \\ 1 + \log \frac{x}{z}, & \frac{1}{r} = 1 - \beta, \\ \left(\frac{z}{x}\right)^{(\beta-1)r+1}, & \frac{1}{r} < 1 - \beta. \end{cases}$$

Since $I_4 \simeq x^{-(2\alpha+1)r} x^{\rho+1-r}$, we see that $I_4 \lesssim I_3$. Moreover, as easily verified, $I_1 + I_2 \lesssim I_3$. Thus the estimate of the theorem follows for $x \gg z$ and by symmetry also for $x \ll z$.

Case 3. $\alpha + \beta = 1/2$. To treat I_1, we notice that on the interval of integration $(x+z)^2 - t^2 \simeq xz$, and hence the logarithm can be neglected, and the estimates for I_1 are as in Cases 1 and 2. Dealing with I_2, we take into account that $t \simeq x + z$ on the interval of integration and get

$$I_2 \simeq (xz)^{-(\alpha+1/2)r}(x+z)^{\rho-r-1} \int_{\sqrt{x^2+z^2}}^{x+z} \log^r \left(\frac{8xz}{(x+z)^2 - t^2}\right) t \, dt$$

$$\simeq (xz)^{-(\alpha+1/2)r+1}(x+z)^{\rho-r-1} \int_0^{1/4} \log^r \frac{1}{w} \, dw.$$

Here the second relation is obtained by changing the variable $[(x+z)^2 - t^2]/(8xz) = w$, and the last integral is a (positive and finite) constant depending only on r. In I_3, we still have $t \simeq x + z$, and thus

$$I_3 \simeq (x+z)^{\rho-r-1} \int_{x+z}^{\sqrt{2}(x+z)} \left[t^2 - (x-z)^2\right]^{(\beta-1)r} \log^r \left(2 \frac{t^2 - (x-z)^2}{t^2 - (x+z)^2}\right) t \, dt$$

$$= 2(xz)^{-(\alpha+1/2)r+1}(x+z)^{\rho-r-1} \int_0^{\frac{(x+z)^2}{(x+z)^2+4xz}} (1-w)^{(1-\beta)r-2} \log^r \frac{2}{w} \, dw,$$

where the last identity follows by the change of variable (5.8) and the equality $\beta - 1 = -(\alpha + 1/2)$. Observe that in the last integral the logarithmic factor can be neglected for our purposes, because it is integrable near 0, and the upper limit of integration is always separated from 0; thus Lemma 5.2(b) is applicable. Finally, in I_4, $t^2 - (x-z)^2 \simeq t^2 - (x+z)^2 \simeq t^2$, so its behavior is the same as in Case 2.

From here we proceed similarly as in Cases 1 and 2 to see that I_1 is the dominating integral when $x \simeq z$, whereas for noncomparable x and z the dominating one is I_3. Then combining the behaviors of I_1 and I_3 we conclude the desired estimate in Case 3.

Proving the bound of Theorem 5.1 is accomplished in the most involved situation when estimates of items (2c) and (3d) in Theorem 3.3 are combined. Other combinations of items (2a)–(2c) and (3a)–(3d) are implicitly contained in the analysis thus

far, since the other bounds coincide with subcases occurring in (2c) and (3d). Further details are straightforward and thus omitted.

Tracing this proof reveals that the conditions (5.1) are indeed necessary to assure integrability in various places. If either of them would not be satisfied, then we would have $I_1 + I_2 + I_3 + I_4 = \infty$. Moreover, since the bounds of Theorem 3.3 are sharp when $t \to |x - z|$ or $t \to x + z$ or $t \to \infty$, we infer that the $L^r(t^\rho dt)$ norm of the kernel is infinite if (5.1) does not hold.

Finally, conditions (a)–(c) allowing us to replace $\lesssim$ by $\simeq$ are deduced from the corresponding comments in Theorem 3.3 and mutual relations between the integrals $I_1, \ldots, I_4$. More precisely, for showing (b) and (c) the following fact is relevant: assuming $-\beta \notin \mathbb{N}$, the sum $I_1 + \cdots + I_4$ is controlled by $I_3 + I_4$ when x and z stay noncomparable or $r < \rho + 1$.   □

6. Mixed norm estimates for $M_t^{\alpha,\beta}$. In this section our aim is to study boundedness of $M_t^{\alpha,\beta}$ from $L^p(d\mu_\alpha)$ to the mixed norm space $L^q(L_{t^\rho}^r)(d\mu_\alpha)$. More generally, we are interested in two-weight mixed norm estimates of the form

$$(6.1) \qquad \left\| \left\| M_t^{\alpha,\beta} f(x) \right\|_{L^r(t^\rho dt)} x^{-B} \right\|_{L^q(d\mu_\alpha)} \lesssim \left\| f(x) x^A \right\|_{L^p(d\mu_\alpha)}$$

which are uniform in f. Our objective is to find possibly wide ranges of the parameters $\alpha, \beta, A, B, r, \rho, p, q$ for which (6.1) holds. Here, in general, we consider

$$(6.2) \qquad \alpha > -1, \quad \beta > -\alpha - 1/2, \quad 1 \le p, q \le \infty, \quad 1 \le r < \infty, \quad A, B, \rho \in \mathbb{R}.$$

The result below is a simple consequence of homogeneity of the kernel $K_t^{\alpha,\beta}(x, z)$; see (1.2).

PROPOSITION 6.1. *Assume that the parameters satisfy* (6.2). *Then the condition*

$$(6.3) \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{2\alpha + 2}\left(A + B - \frac{\rho + 1}{r} \right)$$

is necessary for (6.1) *to hold uniformly in, say,* $f \in C_c^\infty(0, \infty)$.

Notice that condition (6.3) is independent of β.

To proceed, we shall consider a positive kernel $K_{r,\rho}^{\alpha,\beta}(x, z)$ defined by the right-hand side of the bound from Theorem 5.1, without assuming (5.1), and will analyze the corresponding positive operator $K_{r,\rho}^{\alpha,\beta}$.

6.1. Analysis of the auxiliary operator $K_{r,\rho}^{\alpha,\beta}$. Let

$$K_{r,\rho}^{\alpha,\beta}(x, z) = (x + z)^{-2\alpha - 1} \begin{cases} |x - z|^{(\rho+1)/r - 1}, & \frac{\rho+1}{r} < 1 \\ 1 + \log^{1/r}\left(\frac{x+z}{|x-z|}\right), & \frac{\rho+1}{r} = 1 \\ (x + z)^{(\rho+1)/r - 1}, & \frac{\rho+1}{r} > 1 \end{cases}$$

$$\times \begin{cases} 1, & \beta + \frac{1}{r} > 1 \\ 1 + \chi_{\{\beta \neq 0\}} \log^{1/r}\left(\frac{x}{z} \vee \frac{z}{x}\right), & \beta + \frac{1}{r} = 1 \\ \left(\frac{x}{z} \wedge \frac{z}{x}\right)^{\beta + 1/r - 1}, & \beta + \frac{1}{r} < 1 \end{cases},$$

and consider the associated integral operator

$$K_{r,\rho}^{\alpha,\beta} f(x) = \int_0^\infty K_{r,\rho}^{\alpha,\beta}(x, z) f(z)\, d\mu_\alpha(z).$$

We denote by $\operatorname{Dom} K_{r,\rho}^{\alpha,\beta}$ the natural domain of this operator, that is, the set of all functions f for which the defining integral converges for a.a. x. Observe that a necessary condition for $\operatorname{Dom} K_{r,\rho}^{\alpha,\beta}$ to be nontrivial is $\rho > -1$, since otherwise for each $x > 0$ the factor $|x - z|^{(\rho+1)/r-1}$ is not locally integrable around x.

We will show the following sharp result.

THEOREM 6.2. *Assume that* $\alpha, \beta, A, B, r, \rho, p, q$ *are as in* (6.2).
(i) *The inclusion* $L^p(x^{Ap}d\mu_\alpha) \subset \operatorname{Dom} K_{r,\rho}^{\alpha,\beta}$ *holds if and only if* $\rho > -1$ *and*

(6.4)
$$\frac{\rho+1}{r} - \frac{2\alpha+2}{p} - \left(\beta + \frac{1}{r} - 1\right) \wedge 0 < A < \frac{2\alpha+2}{p'} + \left(\beta + \frac{1}{r} - 1\right) \wedge 0$$
$$\left(both \ \leq \ when \ p = 1 \ and \ \left[\beta = 0 \ or \ \beta + \frac{1}{r} - 1 \neq 0\right]\right).$$

(ii) *The estimate*
$$\left\|x^{-B} K_{r,\rho}^{\alpha,\beta} f\right\|_{L^q(d\mu_\alpha)} \lesssim \left\|x^A f\right\|_{L^p(d\mu_\alpha)}$$

holds uniformly in $f \in L^p(x^{Ap}d\mu_\alpha)$ *if and only if the following conditions are satisfied:*
(C1) $p \leq q$;
(C2) $\frac{1}{q} = \frac{1}{p} + \frac{1}{2\alpha+2}\left(A + B - \frac{\rho+1}{r}\right)$;
(C3) $\left(A - \frac{2\alpha+2}{p'}\right) \vee \left(B - \frac{2\alpha+2}{q}\right) < \left(\beta + \frac{1}{r} - 1\right) \wedge 0$
$\qquad$ ($\leq$ *when* $p = q' = 1$ *and* $[\beta = 0$ *or* $\beta + \frac{1}{r} \neq 1]$);
(C4) $\frac{1}{q} \geq \frac{1}{p} - \frac{\rho+1}{r}$ (> *when* $p = 1$ *or* $q = \infty$).

Note that in general none of conditions (C1)–(C4) follows from the others. However, (C4) is superfluous when $\frac{\rho+1}{r} > 1$, and in case $\frac{\rho+1}{r} = 1$ it is equivalent to $(p,q) \neq (1,\infty)$. Moreover, in view of (C2), condition (C4) can be replaced by
(C4') $A + B \geq (2\alpha + 1)\left(\frac{1}{q} - \frac{1}{p}\right)$ (> *when* $p = 1$ *or* $q = \infty$).
Finally, notice that (C2) is exactly (6.3) from Proposition 6.1.

In order to prove Theorem 6.2 we now define auxiliary positive operators into which $K_{r,\rho}^{\alpha,\beta}$ will be "decomposed." Observe that

$$K_{r,\rho}^{\alpha,\beta}(x,z)$$
$$\simeq (x+z)^{-2\alpha-1}(x+z)^{(\rho+1)/r-1}\begin{cases} 1, & \beta + \frac{1}{r} > 1 \\ 1 + \chi_{\{\beta \neq 0\}}\log^{1/r}\left(\frac{x}{z} \vee \frac{z}{x}\right), & \beta + \frac{1}{r} = 1 \\ \left(\frac{x}{z} \wedge \frac{z}{x}\right)^{\beta+1/r-1}, & \beta + \frac{1}{r} < 1 \end{cases}$$
$$+ \chi_{\{x/2 < z < 2x\}} z^{-2\alpha-1}\left[\chi_{\{\frac{\rho+1}{r}<1\}}|x-z|^{\frac{\rho+1}{r}-1} + \chi_{\{\frac{\rho+1}{r}=1\}}\log^{\frac{1}{r}}\frac{x+z}{|x-z|}\right]$$

uniformly in $x, z > 0$. Accordingly, for $\eta \in \mathbb{R}$, define

$$H_0^\eta f(x) = x^{-2\alpha-2+(\rho+1)/r-\eta} \int_0^x z^{2\alpha+1+\eta} f(z) \, dz,$$

$$H_\infty^\eta f(x) = x^\eta \int_x^\infty z^{(\rho+1)/r-1-\eta} f(z) \, dz,$$

$$H_0^{\log} f(x) = x^{-2\alpha-2+(\rho+1)/r} \int_0^x \log^{1/r}\left(\frac{2x}{z}\right) z^{2\alpha+1} f(z) \, dz,$$

$$H_\infty^{\log} f(x) = \int_x^\infty \log^{1/r}\left(\frac{2z}{x}\right) z^{(\rho+1)/r-1} f(z)\, dz,$$

$$Tf(x) = \int_{x/2}^{2x} |x-z|^{(\rho+1)/r-1} f(z)\, dz,$$

$$Sf(x) = \int_{x/2}^{2x} \log^{1/r}\left(\frac{x+z}{|x-z|}\right) f(z)\, dz.$$

Then the following relation is uniform both pointwise and in $f \geq 0$:

$$K_{r,\rho}^{\alpha,\beta} f \simeq H_0^{\beta+1/r-1} f + H_\infty^{\beta+1/r-1} f + \chi_{\{\beta+1/r>1\}}(H_0^0 f + H_\infty^0 f)$$
$$+ \chi_{\{\beta+1/r=1,\beta\neq0\}}(H_0^{\log} f + H_\infty^{\log} f) + \chi_{\{\frac{\rho+1}{r}<1\}} Tf + \chi_{\{\frac{\rho+1}{r}=1\}} Sf.$$

It is clear that $K_{r,\rho}^{\alpha,\beta}$ is bounded from $L^p(x^{Ap} d\mu_\alpha)$ to $L^q(x^{-Bq} d\mu_\alpha)$ (or well defined on $L^p(d\mu_\alpha)$) if and only if all of the component operators appearing on the right-hand side above have the property. Therefore we now analyze each of these operators. We will argue similarly as in [13, section 4.1]. Our main tool will be the following characterization of two power-weight $L^p - L^q$ inequalities for the Hardy operator and its dual; see, e.g., [2, 18] and [13, Lemma 4.1].

LEMMA 6.3. *Let $a, b \in \mathbb{R}$, and let $1 \leq p, q \leq \infty$.*
(a) *The estimate*

$$\left\| x^b \int_0^x g(y)\, dy \right\|_{L^q(\mathbb{R}_+, dx)} \lesssim \|x^a g\|_{L^p(\mathbb{R}_+, dx)}$$

holds uniformly in $g \in L^p(\mathbb{R}_+, x^{ap} dx)$ if and only if $p \leq q$ and $a - \frac{1}{p'} = b + \frac{1}{q}$ and $a < \frac{1}{p'}$ ($\leq$ in case $p = q' = 1$).
(b) *The estimate*

$$\left\| x^b \int_x^\infty g(y)\, dy \right\|_{L^q(\mathbb{R}_+, dx)} \lesssim \|x^a g\|_{L^p(\mathbb{R}_+, dx)}$$

holds uniformly in $g \in L^p(\mathbb{R}_+, x^{ap} dx)$ if and only if $p \leq q$ and $a - \frac{1}{p'} = b + \frac{1}{q}$ and $b > -\frac{1}{q}$ ($\geq$ in case $p = q' = 1$).

For further reference, we state the following conditions:
(C5) $A < \frac{2\alpha+2}{p'} + \left(\beta + \frac{1}{r} - 1\right) \wedge 0$ ($\leq$ when $p = q' = 1$ and $[\beta = 0$ or $\beta + \frac{1}{r} \neq 1]$);
(C5a) $A < \frac{2\alpha+2}{p'} + \beta + \frac{1}{r} - 1$ ($\leq$ when $p = q' = 1$);
(C5b) $A < \frac{2\alpha+2}{p'}$ ($\leq$ when $p = q' = 1$);
(C5c) $A < \frac{2\alpha+2}{p'}$;
(C6) $B < \frac{2\alpha+2}{q} + \left(\beta + \frac{1}{r} - 1\right) \wedge 0$ ($\leq$ when $p = q' = 1$ and $[\beta = 0$ or $\beta + \frac{1}{r} \neq 1]$);
(C6a) $B < \frac{2\alpha+2}{q} + \beta + \frac{1}{r} - 1$ ($\leq$ when $p = q' = 1$);
(C6b) $B < \frac{2\alpha+2}{q}$ ($\leq$ when $p = q' = 1$);
(C6c) $B < \frac{2\alpha+2}{q}$.

Notice that (C5) and (C6) together are equivalent to (C3) from Theorem 6.2.

Analysis of $H_0^{\beta+1/r-1}$. Substituting $f(z) = z^{-2\alpha-\beta-1/r} g(z)$ we see that the estimate

$$\left\| x^{-B} H_0^{\beta+1/r-1} f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

is equivalent to

$$\left\| x^{-2\alpha-\beta+\rho/r-1-B+(2\alpha+1)/q} \int_0^x g(z)\,dz \right\|_{L^q(\mathbb{R}_+,dx)}$$
$$\lesssim \left\| x^{-2\alpha-\beta-1/r+A+(2\alpha+1)/p} g \right\|_{L^p(\mathbb{R}_+,dx)}.$$

By Lemma 6.3(a), this holds if and only if (C1), (C2), and (C5a) hold simultaneously.

Analysis of $H_\infty^{\beta+1/r-1}$. Substituting $f(z) = z^{\beta-\rho/r}g(z)$ we can write the estimate

$$\left\| x^{-B} H_\infty^{\beta+1/r-1} f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

in the equivalent form

$$\left\| x^{\beta+1/r-1-B+(2\alpha+1)/q} \int_x^\infty g(z)\,dz \right\|_{L^q(\mathbb{R}_+,dx)} \lesssim \left\| x^{\beta-\rho/r+A+(2\alpha+1)/p} g \right\|_{L^p(\mathbb{R}_+,dx)}.$$

Applying Lemma 6.3(b), we see that this holds if and only if (C1), (C2), and (C6a) hold simultaneously.

Clearly, this analysis of $H_0^{\beta+1/r-1}$ and $H_\infty^{\beta+1/r-1}$ is valid for any real $\eta = \beta + 1/r - 1$.

Analysis of H_0^0 *and* H_∞^0. This is a special case of the above, so we infer that

$$\left\| x^{-B} H_0^0 f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

holds if and only if (C1), (C2), and (C5b) are satisfied. Further,

$$\left\| x^{-B} H_\infty^0 f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

holds if and only if (C1), (C2), and (C6b) are satisfied.

Analysis of $H_0^{\log}$ *and* $H_\infty^{\log}$. We observe that, given any $\eta < 0$,

$$(6.5) \qquad H_0^0 f \lesssim H_0^{\log} f \lesssim H_0^\eta f \quad \text{and} \quad H_\infty^0 f \lesssim H_\infty^{\log} f \lesssim H_\infty^\eta f, \qquad f \geq 0.$$

This implies that conditions for boundedness of the logarithmic operators are (C1), (C2), and either (C5a) or (C6a), with $\eta = \beta + 1/r - 1 = 0$, but excluding the case $p = q' = 1$ in which weak inequality appears in (C5a) and (C6a). Thus, assuming for the moment that $(p,q) \neq (1,\infty)$, we see that

$$(6.6) \qquad \left\| x^{-B} H_0^{\log} f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

holds if and only if (C1), (C2), and (C5c) are satisfied, and similarly

$$(6.7) \qquad \left\| x^{-B} H_\infty^{\log} f \right\|_{L^q(d\mu_\alpha)} \lesssim \|x^A f\|_{L^p(d\mu_\alpha)}$$

holds if and only if (C1), (C2), and (C6c) are satisfied.

The remaining case requires further treatment. Assuming (C1), (C2) and that $(p,q) = (1,\infty)$, it is seen directly that (6.6) does not hold when $A = \frac{2\alpha+2}{p'} = 0$ and, similarly, (6.7) is not true if $B = \frac{2\alpha+2}{q} = 0$. So, in general, here the boundedness conditions are (C1), (C2), and either (C5c) or (C6c), respectively.

Analysis of T *in case* $\frac{\rho+1}{r} < 1$ *and* S *in case* $\frac{\rho+1}{r} = 1$. Here we may assume that (C1) and (C2) are satisfied, and $\rho > -1$. We can then invoke the analysis of

T and S performed in [13, section 4.1], with the quantity $\frac{\rho+1}{2r}$ playing the role of σ from [13]. To be precise, here the bound $\frac{\rho+1}{2r} < \alpha + 1$ may not be satisfied, but that does not affect the arguments in question. Thus the conclusion is that (C4) from Theorem 6.2 is necessary and sufficient (under the assumptions made) both for T and S, separately, to be bounded from $L^p(x^{Ap}d\mu_\alpha)$ to $L^q(x^{-Bq}d\mu_\alpha)$.

We can finally prove the theorem.

Proof of Theorem 6.2. To prove (i), we proceed as in the proof of [13, Theorem 2.5(i)]. As there, we know that T is well defined on $L^p(x^{Ap}d\mu_\alpha)$ whenever $\sigma = \frac{\rho+1}{2r} > 0$, and S is always well defined on $L^p(x^{Ap}d\mu_\alpha)$. So one has to look at the Hardy-type operators $H_0^\eta, H_\infty^\eta, H_0^{\log}, H_\infty^{\log}$.

Arguing as in [13] we find that the condition

$$\frac{\rho+1}{r} - \frac{2\alpha+2}{p} - \eta < A < \frac{2\alpha+2}{p'} + \eta \qquad (\text{both } \leq \text{ if } p = 1)$$

is necessary and sufficient for the sum $H_0^\eta + H_\infty^\eta$ to be well defined on $L^p(x^{Ap}d\mu_\alpha)$. Further, using (6.5), we infer that the same condition with $\eta = 0$ is necessary and sufficient for the sum $H_0^{\log} + H_\infty^{\log}$ to be well defined on $L^p(x^{Ap}d\mu_\alpha)$, but now without weakening the inequalities in case $p = 1$. The latter is easily verified by means of suitable counterexamples. Summing up,

$$H_0^{\beta+1/r-1} + H_\infty^{\beta+1/r-1} + \chi_{\{\beta+1/r>1\}}(H_0^0 + H_\infty^0) + \chi_{\{\beta+1/r=1,\beta\neq0\}}(H_0^{\log} + H_\infty^{\log})$$

is well defined on $L^p(x^{Ap}d\mu_\alpha)$ if and only if (6.4) holds. The conclusion follows.

The proof of (ii) is essentially contained in the analysis of $H_0^{\beta+1/r-1}$, $H_\infty^{\beta+1/r-1}$, H_0^0, H_∞^0, $H_0^{\log}$, $H_\infty^{\log}$, T, S done above. Observe that (C5a), (C5b) when $\beta + 1/r > 1$ and (C5c) when $\beta + 1/r = 1$ and $\beta \neq 0$ altogether are equivalent to (C5). Analogous observations pertain to (C6a), (C6b), (C6c), and (C6). $\square$

Remark 6.4. Perhaps a bit surprisingly, the operator $K_{r,\rho}^{\alpha,\beta}$ much resembles the potential operator related to the modified Hankel transform; see [13, section 2.1]. Even more, if $\sigma = \frac{\rho+1}{2r} < \alpha + 1$ and $[\beta + 1/r - 1 > 0$ or $(\beta, r) = (0, 1)]$, then the kernel $K_{r,\rho}^{\alpha,\beta}(x,z)$ behaves exactly like the potential kernel $K^{\alpha,\sigma}(x,z)$, and one gets the same boundedness results for the associated operators.

We now comment on a shape of the set of all $1 \leq p, q \leq \infty$ for which the estimate of Theorem 6.2 holds. Actually, we are going to look at the corresponding set D of pairs $(\frac{1}{p}, \frac{1}{q})$ in the closed unit square $[0,1]^2$. Then, depending on the parameters involved, the following situations occur (but no others):

(S1) D is a segment in the lower triangle $\frac{1}{p} > \frac{1}{q}$, parallel to the diagonal and with endpoints included and located on the boundary of the square.

(S2) D is a subsegment of that from (S1) having excluded any endpoint not lying on the boundary of the square.

(S3) D is a subsegment of the diagonal of $[0,1]^2$, of length strictly smaller than the diagonal, having excluded any endpoint not lying on the boundary of the square.

(S4) D is just one point, the lower-right vertex of the square.

(S5) D is empty.

All segments of types (S1)–(S3) indeed occur with suitable choices of the parameters.

6.2. Main results. By homogeneity of the kernel (1.2) it follows that a power-weighted space $L^p(x^{Ap}d\mu_\alpha)$ is included in Dom $M_t^{\alpha,\beta}$ for a given $t > 0$ if and only if

$L^p(x^{Ap}d\mu_\alpha) \subset \text{Dom } M_t^{\alpha,\beta}$ for all $t > 0$. Thus from Theorems 5.1 and 6.2(i) we conclude that all the weighted $L^p(d\mu_\alpha)$ spaces admitted in Theorem 6.2 (i) are contained, under all the relevant assumptions on the parameters, in $\text{Dom } M_t^{\alpha,\beta}$, for all $t > 0$.

The main result of this paper is a straightforward consequence of Theorems 5.1 and 6.2. It reads as follows.

THEOREM 6.5. *Let* $\alpha > -1$, $\beta > -\alpha - 1/2$, $1 \le p, q \le \infty$, $1 \le r < \infty$, $A, B, \rho \in \mathbb{R}$, *and assume that*

$$\alpha + \beta > \frac{1}{2} - \frac{1}{r} \qquad and \qquad \left[\frac{\rho+1}{r} < 2\alpha + 2 \ \ if \ -\beta \notin \mathbb{N}\right].$$

Then, under conditions (C1)–(C4), $L^p(x^{Ap}d\mu_\alpha) \subset \text{Dom } M_t^{\alpha,\beta}$ *for each* $t > 0$, *and the estimate*

$$\left\|\left\|M_t^{\alpha,\beta}f(x)\right\|_{L^r(t^\rho dt)}x^{-B}\right\|_{L^q(d\mu_\alpha)} \lesssim \left\|f(x)x^A\right\|_{L^p(d\mu_\alpha)}$$

holds uniformly in $f \in L^p(x^{Ap}d\mu_\alpha)$.

Remark 6.6. The order of taking the norms in the mixed norm expression in Theorem 6.5 can be exchanged. Indeed, in view of Minkowski's integral inequality, when $q \le r$,

$$\left\|\left\|M_t^{\alpha,\beta}f(x)x^{-B}\right\|_{L^q(d\mu_\alpha)}\right\|_{L^r(t^\rho dt)} \le \left\|\left\|M_t^{\alpha,\beta}f(x)\right\|_{L^r(t^\rho dt)}x^{-B}\right\|_{L^q(d\mu_\alpha)}.$$

From Theorem 6.5 we conclude immediately the following two-weight mixed norm estimate for the generalized spherical mean Radon transform M^β in $\mathbb{R}^n$, $n \ge 1$.

COROLLARY 6.7. *Let* $n \ge 1$. *Then, under the assumptions and conditions of Theorem 6.5 on the parameters, with* $\alpha = n/2 - 1$, *one has the estimate*

$$\left\|\left\|M^\beta f(x,t)\right\|_{L^r(t^\rho dt)}|x|^{-B}\right\|_{L^q(\mathbb{R}^n,dx)} \lesssim \left\|f(x)|x|^A\right\|_{L^p(\mathbb{R}^n,dx)}, \quad f \in L^p_{rad}(\mathbb{R}^n, |x|^{Ap}dx).$$

Here $M^\beta f(\cdot, t)$ *is understood as the extension given by* $M_t^{\alpha,\beta}$ *of this operator defined initially on* $L^2(\mathbb{R}^n, dx) \cap L^p_{rad}(\mathbb{R}^n, |x|^{Ap}dx)$ *by means of the Fourier transform.*

Theorem 6.5 and Corollary 6.7 are new results, even when specified to an unweighted setting ($A = B = 0$) or to nongeneralized radial spherical means ($\beta = 0$). Moreover, given our strategy of proof, these results are pretty precise (if not sharp, at least in some cases) in view of the sharpness statements in Theorems 3.3 (see also the relevant comment succeeding this theorem) and 5.1, and the optimal result contained in Theorem 6.2.

7. Applications to some PDE problems: weighted Strichartz estimates.
Let $\alpha > -1$, $\beta > -\alpha - 1/2$, and assume that $u(x,t) = cM_t^{\alpha,\beta}f(x)$, $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, is a (weak) solution to a PDE problem with f entering an initial-value condition. Then Theorem 6.5 and Remark 6.6 imply a weighted Strichartz-type estimate

$$\left\|\left\|u(x,t)x^{-B}\right\|_{L^q(d\mu_\alpha)}\right\|_{L^r(t^\rho dt)} \lesssim \left\|f(x)x^A\right\|_{L^p(d\mu_\alpha)}$$

under the assumptions and conditions of Theorem 6.5 and provided that $q \le r$. Moreover, if $u(x,t) = ctM_t^{\alpha,\beta}f(x)$ is the solution, then a similar Strichartz-type estimate holds with the parameter ρ and the corresponding assumptions and constraints adjusted suitably.

When $\alpha = n/2 - 1$, $\beta > -n/2 + 1/2$, $n = 1, 2, \ldots$, and $v(x,t) = cM_t^{\alpha,\beta}f_0(|x|)$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, is a spatially radial (weak) solution to a PDE problem with a radial $f = f_0(|\cdot|)$ entering an initial-value condition, then Theorem 6.5, together with Remark 6.6, establishes the weighted Strichartz-type estimate

$$\left\| \left\| v(x,t)|x|^{-B} \right\|_{L^q(\mathbb{R}^n, dx)} \right\|_{L^r(t^\rho dt)} \lesssim \left\| f(x)|x|^A \right\|_{L^p(\mathbb{R}^n, dx)},$$

provided that the parameters satisfy all the restrictions imposed by Theorem 6.5 and $q \leq r$. If $v(x,t) = ctM_t^{\alpha,\beta}f_0(|x|)$ happens to be such a solution, then again one infers a Strichartz-type estimate by taking $\tilde{\rho} = \rho + r$ instead of ρ.

We now give examples of Cauchy initial-value problems for several classical PDEs, where solutions $u(x,t)$ or $v(x,t)$ of the above form indeed occur:

(I) *EPD equation.* Let $n \geq 1$ and $\square_\beta$ be the EPD operator related to $\mathbb{R}^n$,

$$\square_\beta v = \Delta_x v - v_{tt} - \frac{n + 2\beta - 1}{t} v_t.$$

Consider the Cauchy problem in $\mathbb{R}^n \times \mathbb{R}_+$,

(7.1) $\square_\beta v = 0,$ $v(x,0) = f(x),$ $v_t(x,0) = 0,$

with a radial initial position $f = f_0(|\cdot|)$. Then, assuming that $\beta > -n/2 + 1/2$,

$$v(x,t) = M_t^{n/2-1,\beta}f_0(|x|)$$

is a solution to the singular Cauchy problem (7.1); see [22, 3, 17]. Note that here the special case $\beta = 2/3 - n/2$ corresponds to the Tricomi equation.

(II) *Wave equation.* Let $n \geq 1$, and observe that $\square_{(1-n)/2}$ is the wave operator related to $\mathbb{R}^n$. Consider the Cauchy problem in $\mathbb{R}^n \times \mathbb{R}_+$,

(7.2) $\square_{(1-n)/2}v = 0,$ $v(x,0) = 0,$ $v_t(x,0) = f(x),$

with radial initial speed $f = f_0(|\cdot|)$. Then (see [19])

$$v(x,t) = tM_t^{n/2-1,(3-n)/2}f_0(|x|)$$

is a solution to (7.2).

(III) *Bessel EPD and wave equations.* For $\alpha > -1$, let L_α be the one-dimensional Bessel operator

$$L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x}\frac{d}{dx}.$$

When $\alpha = n/2 - 1$, $n = 1, 2, \ldots$, L_α is the radial part of the standard Laplacian in $\mathbb{R}^n$. Let us consider the following differential problems in $\mathbb{R}_+ \times \mathbb{R}_+$:

(7.3) $L_\alpha u - u_{tt} - \dfrac{2\alpha + 2\beta + 1}{t}u_t = 0,$ $u(x,0) = f(x),$ $u_t(x,0) = 0,$

(7.4) $L_\alpha u - u_{tt} = 0,$ $u(x,0) = 0,$ $u_t(x,0) = f(x).$

Then, for $\beta > -\alpha - 1/2$,

$$u(x,t) = M_t^{\alpha,\beta}f(x)$$

is a solution to (7.3), while

$$u(x,t) = tM_t^{\alpha,-\alpha+1/2}f(x)$$

is a solution to (7.4). Observe that the Bessel EPD operator appearing in (7.3) is in fact a difference of two Bessel operators, one of them acting on the spatial variable and the other on the time variable.

It is worth pointing out that solutions to (7.2) and (7.4) with the initial conditions reversed, i.e., when the initial speed is zero and the initial position is prescribed, are expressed as

$$v(x,t) = \mathcal{M}_t^{n/2-1,(1-n)/2}f(x),$$
$$u(x,t) = t\mathcal{M}_t^{\alpha,-\alpha-1/2}f(x).$$

The parameters here, however, correspond to the critical line $\alpha + \beta = -1/2$, where $\mathcal{M}_t^{\alpha,\beta}$ becomes a singular integral, a more subtle case that is not treated in this paper.

Another comment concerns connections of the above mentioned solutions with initial positions/speeds. More precisely, the question explores in what sense the solutions converge to initial conditions as time decreases to 0, and for what ranges of the parameters the convergence takes place. In general, without requiring much regularity of the initial data, this is a difficult question that requires studying time-maximal operators. We refer the reader to [19, 12, 4, 6] for some partial results.

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